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Wave propagation in a one-dimensional poroelastic column

In Biot's theory of porous media a second compressional wave, known as the slow wave, has been identified. An analytical solution in the Laplace transform domain is obtained showing clearly two compressional waves. For the special case of an inviscid fluid, a closed form exact solution in time domain is obtained using an analytical inverse Laplace transform. For the general case of a viscous fluid, solution in time domain is evaluated using the Convolution Quadrature Method of Lubich. Using properties of two different real materials, the wave propagating behavior, in terms of stress, pore pressure, displacement, and flux, are examined. Of most interest is the identification of second compressional wave and its sensitivity of material parameters.

1. Analytical solution

For a wide range of fluid infiltrated materials, such as water saturated soils, oil impregnated rocks, or air filled foams, Biot's theory of poroelasticity is used. Among the significant findings in this theory was the identification of three waves for a 3-d continuum, two compressional waves and one shear wave. This extra compressional wave, known as the slow wave, will here be confirmed numerically.

A 1-d poroelastic column of length ℓ is considered. It is assumed that the side walls and the bottom are rigid, frictionless and impermeable. Hence, the displacements normal to the surface are blocked and the column is otherwise free to slide parallel to the wall. At the top, the total stress σ and the pore pressure p and at the bottom the longitudinal displacement u and the flux q are prescribed. Due to these restrictions the governing set of differential equations is reduced to two scalar coupled ordinary differential equations in Laplace domain (denoted by $\hat{()}$ with parameter s)

$$E\hat{u}_{,xx} - (\alpha-\beta)\,\hat{p}_{,x} - s^2\left(\varrho - \beta\varrho_f\right)\hat{u} = 0 \qquad \frac{\beta}{s\varrho_f}\hat{p}_{,xx} - \frac{\phi^2 s}{R}\hat{p} - (\alpha-\beta)\,s\hat{u}_{,x} = 0\;, \tag{1}$$

with the abbreviation $\beta = \frac{\phi^2 s\kappa\varrho_f}{\phi^2 + s\kappa(\varrho_a + \phi\varrho_f)}$, the modulus E, the porosity ϕ, the bulk density $\varrho = \varrho_s(1-\phi) + \phi\varrho_f$ and the permeability κ. The apparent mass density ϱ_a is assumed to be frequency independent as $\varrho_a = 0.66\phi\varrho_f$. Biot's effective stress coefficient α and R complete the set of material parameters.

Due to the neglected body forces this is a system of homogeneous ordinary differential equations with inhomogeneous boundary conditions. Such a system can be solved by the exponential ansatz $\hat{u}(x) = Ue^{\lambda sx}$, $\hat{p}(x) = Pe^{\lambda sx}$. This leads to an Eigenvalue problem where the characteristic equation has four complex roots, and, therefore, the complete solution of the homogeneous problem is $\hat{u}(x) = \sum_{i=1}^{4} U_i e^{\lambda_i sx}$, $\hat{p}(x) = \sum_{i=1}^{4} P_i e^{\lambda_i sx}$. Using the Eigenvectors a system of four equations for four unknowns is achieved. Finally, for stress boundary conditions $\hat{\sigma}(x=\ell) = -P_0$ and $\hat{u}(x=0) = \hat{q}(x=0) = \hat{p}(x=\ell) = 0$ the results for the displacement and the pressure is obtained

$$\hat{u}(s,x) = \frac{P_0}{E(d_1\lambda_3 - d_3\lambda_1)}\left[\frac{d_3\left(e^{-\lambda_1 s(\ell-x)} - e^{-\lambda_1 s(\ell+x)}\right)}{s\left(1+e^{-2\lambda_1 s\ell}\right)} - \frac{d_1\left(e^{-\lambda_3 s(\ell-x)} - e^{-\lambda_3 s(\ell+x)}\right)}{s\left(1+e^{-2\lambda_3 s\ell}\right)}\right] \tag{2}$$

$$\hat{p}(s,x) = \frac{P_0 d_1 d_3}{E(d_1\lambda_3 - d_3\lambda_1)}\left[\frac{\left(e^{-\lambda_1 s(\ell-x)} + e^{-\lambda_1 s(\ell+x)}\right)}{1+e^{-2\lambda_1 s\ell}} - \frac{\left(e^{-\lambda_3 s(\ell-x)} + e^{-\lambda_3 s(\ell+x)}\right)}{1+e^{-2\lambda_3 s\ell}}\right]. \tag{3}$$

Note, due to the dependence of β to the Laplace parameter s, the roots λ_i and consequently $d_i = \frac{E\lambda_i^2 - (\varrho - \beta\varrho_f)}{(\alpha-\beta)\lambda_i}$ are dependent of s. Therefore, an analytical inverse Laplace transform of the solutions above is in general not possible. However, if the damping due to the relative motion of the fluid and the solid is neglected, i.e. the permeability tends to infinity $\kappa \to \infty \implies \beta \approx \frac{\varphi^2\varrho_f}{\varrho_a + \varphi\varrho_f}$, an analytical inverse Laplace transform can be found [2]. For an arbitrary value of κ a numerical inverse Laplace transformation is necessary. Here, it is preferable to take the 'Convolution Quadrature Method' proposed by Lubich [1]. This method approximates a convolution integral numerically by a quadrature formula whose weights are determined with the help of the Laplace transformed impulse response functions $\hat{u}(s,x)$ and a linear multistep method.

2. Results

Next, wave propagation in the 1-d column is studied using the developed solutions. Two very different materials, a rock (Berea sandstone) and a soil (coarse sand) are chosen to represent a wide range of porous materials. To unambiguously capture the slow wave, we examine an 'infinite' column to avoid wave reflections. This is achieved by using a column length of $\ell = 1000\ m$ and a short observation time. In Fig. 1 we record the pressure, $p\,(t, x = 995\ m)$, five meters behind the excitation point ($x = \ell = 1000\ m$). It is assumed that the time history of the stress loading is a Heaviside step function. Since this is the first time that we expect to observe such wave, it is compared with the

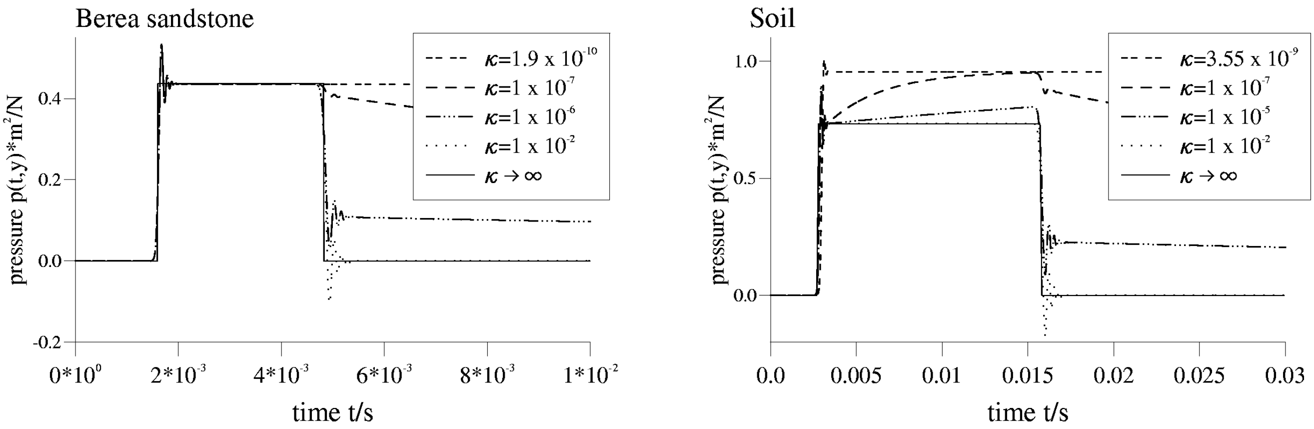

Figure 1: Pressure $p\,(t, x = 995m)$ versus time for different values of κ compared with the analytical solution

exact time domain solution for $\kappa \to \infty$ [2], shown as solid lines in Fig. 1 for the two materials, to gain confidence. To make the comparison, an arbitrarily large value, $\kappa = 1 \times 10^{-2}$, is chosen in the Convolution Quadrature solution, with results plotted in dashed lines in Fig. 1. It is observed that, except for some fluctuations at wave fronts, which are generally unavoidable for all numerical inversion methods, the two solutions compare very well.

The phenomenon exhibited in Fig. 1 can be rationalized as follows. We first observe the arrival of the first wave at 5 m that causes the step jump. The second wave, arriving at a later time, is of negative amplitude and cancels exactly the first wave as indicated by the exact solution. The arrival time of the two waves is independent of κ as its limit has been taken.

To obtain and understand the solution of the realistic cases, we start to decrease κ values. Fig. 1 shows a sequence of reduction that lead to the real values. As κ decreases, we observe that both the amplitude and the arrival time of the waves are affected. The effect is strongest for the second wave. For some intermediate values of κ, we observe that when the second wave arrives, its amplitude is diminished. Hence the pressure does not drop to zero at the passage of the wave front. We also observe that the second wave is dispersive as it does not arrive as a sharp front with constant value in some cases. Rather, the pressure continues to decline as seen in some curves. Also, the comparison shows the different behavior of the two different materials on changing the permeability. For rock, the wave amplitude of the first wave is nearly independent from the permeability, contrary to the soil.

Summarizing, two compressional waves are clearly identified in the limiting case of an infinite permeability. For some intermediate permeability cases the second wave is also present. However, for the actual permeabilities of the tested materials the effect of the second wave vanishes after a short distance. This paper does not test all practical materials, natural and man-made. There exist some materials, particularly those with small fluid viscosity and large medium permeability, in which the second wave effect can be significant.

3. References

1 Lubich, C.: Convolution Quadrature and Discretized Operational Calculus. I. Numerische Mathematik. **52** (1988), 129–145.

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