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A Poroelastic Mindlin-Plate

The numerical treatment of noise insulation of solid walls has been an object of scientific research for many years. The main noise source is the bending vibration of the walls usually modeled as plates. Generally, walls consist of porous material, for instance concrete or bricks. Therefore, a poroelastic plate theory is necessary. A theory of dynamic poroelasticity was developed by Biot using the solid displacements and the pore pressure as unknowns. After formulating the poroelastic theory for thick plates, Mindlin's theory, a variational principle for this poroelastic thick plate model is developed. This is the basis of a Finite Element formulation.

1. Poroelastic Mindlin plate theory

The combination of a poroelastic constitutive assumption and a plate theory was published by Theodorakopoulos and Beskos [1]. They used the Kirchhoff theory assuming thin plates and neglected any in-plane motion to obtain purely bending vibrations. Here, a poroelastic theory of Biot is used and a Mindlin plate is considered to model moderately thick plates.

Following Biot's approach [2] to model the behavior of porous media, an elastic skeleton with a statistical distribution of interconnected pores saturated by a fluid is considered. One possible representation of poroelastic constitutive equation is obtained using the total stress $\sigma_{ij} = \sigma^s_{ij} + \sigma^f \delta_{ij}$ and the pore pressure p as independent variables. Introducing Biot's effective stress coefficient α and the solid displacement u_i the constitutive equation reads

$$\sigma_{ij} = G(u_{i,j} + u_{j,i}) + \frac{2G\nu}{1-2\nu} u_{k,k}\delta_{ij} - \alpha\delta_{ij}p \qquad \zeta = \alpha u_{k,k} + \frac{\phi^2}{R}p \qquad i,j,k = 1,2,3 \tag{1}$$

with the shear modulus of the solid frame G and Poisson's ratio ν. Additionally to the total stress σ_{ij}, as a second constitutive equation, the variation of fluid volume per unit reference volume ζ is introduced with the material constant R and the porosity ϕ. Next, the balance of momentum for the bulk material must be fulfilled. This dynamic equilibrium is given by

$$\sigma_{ij,j} + F_i = \varrho\omega^2 u_i + \phi\varrho_f\omega^2 v_i \tag{2}$$

with the bulk body force per unit volume F_i, the relative fluid to solid displacement v_i, and the bulk density $\varrho = \varrho_s(1-\phi) + \phi\varrho_f$. The density of the solid and the fluid is denoted by ϱ_s and ϱ_f, respectively. Further, the fluid transport in the interstitial space is modeled with a generalized Darcy's law

$$\phi i\omega v_i = q_i = -\kappa(p_{,i} + \varrho_f\omega^2 u_i \frac{\varrho_a + \phi\varrho_f}{\phi}\omega^2 v_i) \tag{3}$$

where κ denotes the permeability and q_i the specific flux. To describe the interaction between fluid and skeleton the apparent mass density ϱ_a was introduced by Biot [2] .

Aiming at the equation of motion, Darcy's law is rearranged to obtain v_i. Then, the final set of differential equations for the displacement u_i and the pore pressure p is obtained by inserting the constitutive equation in the dynamic equilibrium and continuity equation $i\omega\zeta + q_{i,i} = 0$. The following set of equations describes the behavior of a poroelastic continuum completely $(\beta = \kappa\varrho_f\phi^2\omega^2 / \left[\phi^2\omega^2 + \omega^2\kappa(\varrho_a - \phi\varrho_f)\right])$

$$\hat{\sigma}_{ij,j} + \beta\hat{p}_{,i} + \hat{F}_i + \omega^2(\rho - \beta\rho_f)\hat{u}_i = 0 \qquad \frac{\beta}{i\omega\rho_f}p_{,ii} - \frac{i\omega\phi^2}{R}p - i\omega(\alpha - \beta)u_{i,i} = 0. \tag{4}$$

After formulating the 3-d governing equations, the usual plate assumptions for a Mindlin plate will be taken into account. To ensure that there are no in-plane motions, in-plane forces $F_1 = F_2 = 0$ and in-plane flux $q_1 = q_2 = 0$ will be neglected.

So, following Mindlin's plate theory, the displacements are replaced by

$$u_\alpha(x_1, x_2, x_3) = x_3\psi_\alpha(x_1, x_2) \qquad \alpha, \beta = 1, 2 \qquad u_3(x_1, x_2, x_3) = w(x_1, x_2) \tag{5}$$

with the frequency dependent rotations ψ_α and the frequency dependent deflection w. It is assumed that $\sigma_{33} \ll \sigma_{11}, \sigma_{22} \quad \longrightarrow \quad \sigma_{33} \approx 0$, which is used to eliminate the derivative of the out-of-plane displacement u_3 with respect to the x_3-direction

$$\sigma_{33} = 0 \quad \longrightarrow \quad u_{3,3} = \frac{\alpha(1-2\nu)}{2G((1-\nu)} - \frac{\nu}{1-\nu}\, x_3\psi_{\alpha,\alpha}\ . \tag{6}$$

The above assumptions are inserted in the out-of-plane direction $i = 3$ of equation (4a) and an integration over the thickness from $-h/2$ to $h/2$ is performed yielding the governing equation for the deflection

$$G(\psi_{\alpha,\alpha} + w_{,\alpha\alpha})h_s + \beta\Delta p + \int_{-\frac{h}{2}}^{\frac{h}{2}} F_3 x_3 dx_3 + \omega^2(\rho - \beta\rho_f)hw = 0 \quad \text{with} \quad \Delta p = p_o - p_u. \tag{7}$$

The pore pressure p_u and p_o on the upper side of the plate ($x_3 = h/2$) and on the lower side of the plate $x_3 = -h/2$ are due to the partial integration performed on $\int_{-h/2}^{+h/2} p_{,33}dx_3$ during the above explained integration over the thickness. Next, the equations for the rotations are achieved by using the in-plane directions $i = 1, 2$ of equation (4a). Before integrating them over the thickness, they are multiplicated by x_3. After eliminating $u_{3,3}$ using (6) the equations for the rotations read

$$\frac{h^3}{12}G\psi_{\alpha,\beta\beta} + \frac{h^3}{12}G\frac{1+\nu}{1-\nu}\psi_{\beta,\alpha\beta} - (\alpha\frac{1-2\nu}{1-\nu} - \beta)Q_{,\alpha} + h_sG(\psi_\alpha + w_{,\alpha}) + \omega^2(\rho - \beta\rho_f)\psi_\alpha\frac{h^3}{12} = 0. \tag{8}$$

In equation (8), the abbreviation $Q = \int_{-h/2}^{+h/2} x_3 p(x_1, x_2, x_3)dx_3$ was used. The last step is to perform the same operations also on the equation for the pore pressure, i.e., multiplication with x_3, integration over the thickness, and elimination of $u_{3,3}$. This yields the equation for the integrated pore pressure Q as

$$\frac{\beta}{i\omega\rho_f}Q_{,\alpha\alpha} - i\omega\left[\frac{\phi^2}{R} + (\alpha-\beta)\frac{\alpha(1-2\nu)}{2G(1-\nu)}\right]Q - i\omega(\alpha-\beta)\frac{1-2\nu}{1-\nu}\frac{h^3}{12}\psi_{\alpha,\alpha} + \frac{\beta}{i\omega\rho_f}\left[\frac{h}{2}\Delta q - \Delta p\right] = 0 \tag{9}$$

where Δq is the difference between the normal derivative of the pore pressure on the upper and lower surface of the plate $\Delta q = p_{,3}(h/2) - p_{,3}(-h/2)$. With the equations (7),(8), and (9) sufficient equations for the unknowns deflection w, rotation ψ_α, and integrated pore pressure Q are given. For the pore pressure p an assumption on its distribution over the thickness must be made to calculate the integral Q.

2. Variational principle and FE formulation

To develop a Finite Element Method for the proposed poroelastic plate, first, the principle of virtual work must be formulated.

This is achieved by multiplying the deflection governing equation (7) with the variation of the deflection δw and the rotation governing equation (8) with the variation of the rotation $\delta\psi_\alpha$ and integration of both equations over the surface of the plate A. Here, additionally the third equation for the integrated pore pressure (9) has to be taken into acount. Consequently, the variation of the integrated pore pressure δQ is chosen as weighting function. Finally, all three described parts are summed up and a partial integration yields the principle of virtual work. The explicit expressions are skipped here due to lack of space.

Choosing ansatz function in this principle of virtual work results in a Finite Element formulation. As usual in poroelastic FEM, the pore pressure is approximated with an ansatz order less than the deflection and rotation.

3. References

1 Theodorakopoulos, D.,Beskos, D: Flexural vibrations of poroelasticplates. Acta Mechanica 103, 191-203 (1994).

2 Biot, M.A.: Theory of propagation of elastic waves in a fluid-saturated porous solid I/II:Low/Higher frequency range. J.Acoust.Soc.Am 28,168-191 (1956).