

Modelling of wave mode conversion in fusion plasmas by the resonant layer method *

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Introduction. The problem of efficient modelling of wave mode conversion arises in various areas of fusion plasma theory. Those areas include ICRH and ICCD in a tokamak plasma with two or more sorts of ions, the problem of interaction of low frequency (or static) resonant magnetic field perturbations (RMPs) with the tokamak plasma, etc. In this kind of modelling, besides the large scale waves (“fast modes”) whose wavelength is not extremely small as compared to the size of the device, there exist also wave modes with wavelengths comparable to the ion Larmor radius or even smaller (“slow modes”). The existence of such short scale solutions causes essential difficulties in the numerical modelling of the electromagnetic field interaction with the plasma. Depending on the method, the modelling either requires a very fine grid or a huge number of Fourier modes. As a result, inversions of large and sometimes ill conditioned matrices are needed (problem is numerically “stiff”). Moreover, due to finite Larmor radius (FLR) effects and kinetic effects connected with the parallel motion of resonant particles, the plasma conductivity is non-local, i.e. the Maxwell equations become integro-differential.

The Resonant Layer Method. The presented method uses the fact that the various wave modes are well separated in most of the plasma volume, except inside the conversion zone (“resonant layer”). Therefore, the fast mode oscillations can be effectively modelled by finite difference methods whereas the various short scale slow modes are well described by WKB theory (geometrical optics) nearly everywhere. Thus, the treatment of the “full-wave” problem where all wave modes have to be computed simultaneously is needed only inside the resonant layer where the above mentioned approximations break down. Below, this method is explained for the case

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of mode conversion during ICRH with two sorts of ions in a slab geometry representing the tokamak. Namely, the main magnetic field is given in Cartesian variables as $\mathbf{B}_0 = \mathbf{e}_x B_{0x} + \mathbf{e}_z B_{0z} x_0/x$ where B_{0x} , B_{0z} and x_0 are constant, the plasma is homogeneous and is surrounded by ideally conducting walls at $x = x_1$ and $x = x_2$, and an infinitely thin antenna is located at $x = x_A$. Fourier analysis over y and z variables reduces the problem to 1D.

The general, integro-differential form of Maxwell equations describing the “full-wave” problem can be presented in the form of an exact Maxwell operator acting on the wave electric field,

$$\hat{\mathbf{M}} \cdot \mathbf{E} \equiv \nabla \times (\nabla \times \mathbf{E}) - \frac{\omega^2}{c^2} \mathbf{E} - \frac{4\pi i \omega}{c^2} \hat{\sigma} \cdot \mathbf{E} = 0, \quad (1)$$

where $\hat{\sigma}$ is an integral plasma conductivity operator. The model solutions to (1) which approximately satisfy Maxwell equations outside the resonant layer are denoted as $\mathbf{E}^{(k)}$ where $k = 0, 1, 2, \dots$. It should be noted that the approximate solutions are normally singular at the mode conversion surface (mode conversion point x_c in 1D) either due to the singularity of the simplified equation (fast modes) or due to the violation of the WKB approximation (slow modes). In the model solutions, those singularities are removed by adding finite terms which are strongly localized around the conversion point either to the equation (fast modes) or directly to the dispersion relation (WKB for slow modes). Various model solutions $\mathbf{E}^{(k)}$ differ from each other only by the amplitude of these additional terms a_k . Each of the model solutions $E^{(k)}$ for such a regularized problem is a full solution satisfying boundary conditions at the wall and matching conditions at the antenna and, generally, it necessarily contains both, fast and slow mode fields. None of them, if taken alone, even approximately represents the exact solution for the boundary problem, however, a linear combination

$$\mathbf{E}_{\text{mod}} = \mathbf{E}^{(0)} + \sum_{k=1}^N C_k \Delta \mathbf{E}^{(k)}, \quad \Delta \mathbf{E}^{(k)} = \mathbf{E}^{(k)} - \mathbf{E}^{(0)}, \quad (2)$$

does represent the exact solution almost in the whole volume, except a certain relatively small vicinity $x^- < x < x^+$, $x^\pm = x_c \pm \Delta x$ of the conversion point (“resonant layer”). The number N of model solutions (besides $\mathbf{E}^{(0)}$) is determined by the order of a fast wave problem (e.g., for $B_{0x} = 0$ fast wave is described by a second order ODE) plus twice the number of travelling slow modes (evanescent slow modes are ignored). In order to determine the unknown constants C_k and the correct solution in the boundary layer, the exact problem (1) in the layer is re-formulated for the correction term $\delta \mathbf{E} = \mathbf{E} - \mathbf{E}_{\text{mod}}$,

$$\hat{\mathbf{M}} \cdot \delta \mathbf{E} = -\hat{\mathbf{M}} \cdot \mathbf{E}_{\text{mod}} = \frac{4\pi i \omega}{c^2} \left(\delta \mathbf{j}^{(0)} + \sum_{k=1}^N C_k \left(\delta \mathbf{j}^{(k)} - \delta \mathbf{j}^{(0)} \right) \right). \quad (3)$$

Here, the correction plasma currents $\delta\mathbf{j}^{(k)}$ are fully determined by the model solutions $\mathbf{E}^{(k)}$ and are strongly localized around x_c so that near the boundaries of the resonant layer, where the model solutions satisfy Maxwell equations well enough, $\delta\mathbf{j}^{(k)}$ are negligible small. Thus, $\delta\mathbf{j}^{(k)}$ can be well approximated by a function which, together with all derivatives, is enforced to be zero at the resonant layer boundaries. If Fourier analysis is applied to such a function,

$$\mathbf{j}^{(k)} = \sum_{m=-\infty}^{\infty} \mathbf{j}_m^{(k)} \exp\left(\frac{\pi m(x-x_c)}{\Delta x}\right), \quad (4)$$

the spectrum of this function $\mathbf{j}_m^{(k)}$ converges exponentially with $|m| \rightarrow \infty$. Applying Fourier analysis also for the correction field $\delta\mathbf{E}$, Maxwell equations (3) are reduced to an infinite set of algebraic equations for the Fourier amplitudes of the correction field, $\delta\mathbf{E}_m$, which, due to a fast decay of $\mathbf{j}_m^{(k)}$, can be truncated keeping a rather limited number of Fourier harmonics. The correction field $\delta\mathbf{E}$ obtained in this way is formally a periodic function of x but it need not to be a localized function as long as the model solution \mathbf{E}_{mod} is not close to the exact solution outside and nearby the resonant layer boundaries, $x^\pm - \delta x < x < x^\pm + \delta x$, where $\delta x \ll \Delta x$. In order to localize $\delta\mathbf{E}$ we look for the minimum of the functional

$$S = \int_{x^+ - \delta x}^{x^+ + \delta x} dx |\delta\mathbf{E}(x)|^2 \quad (5)$$

with respect to constants the C_k and obtain a linear equation set for C_k . Thus, both, the proper model solution \mathbf{E}_{mod} and the localized correction term $\Delta\mathbf{E}$ are simultaneously fully defined after the calculation of C_k . Note that the Fourier spectrum of the correction field saturates rapidly only for a proper choice of C_k since otherwise the periodic solution for $\Delta\mathbf{E}$ must represent an aperiodic function whose spectrum decays as slowly as $1/|m|$.

Resonant Layer Method Example. For testing the method, a simplified local plasma conductivity $\hat{\sigma} = (4\pi i)^{-1} \omega (\hat{\epsilon} - 1)$ is used in (1). Namely, the hot tensor for a homogeneous plasma and magnetic field [1] is simplified by ignoring the FLR effects and enforcing the condition $k_{\parallel} = k_z$ even in cases with finite B_{0x} representing the poloidal tokamak field. The resulting “full wave” problem (1) described by the ODE of the 4-th order can be solved then directly by Runge-Kutta integration providing a test solution for benchmarking the method.

The parameters correspond to a deuterium plasma with $T_e = T_i = 3$ keV, $n_e = 5 \cdot 10^{13} \text{ cm}^{-3}$. The fundamental cyclotron resonance point for hydrogen minority ions (with 20% concentration) is located at $x = 200$ cm where $B_0 = 20$ kG and $B_{0x}/B_{0z} = 0.1$. The solution has a harmonic dependence on y and z variables with $k_y = k_z = 0.02 \text{ cm}^{-1}$.

In Fig. 1 the real part of the E_x component is shown for two model solutions $\mathbf{E}^{(k)}$.

One of them (MOD) corresponds to the fast mode (fast magnetosonic wave) computed numerically and another one (WKB) corresponds to the slow mode - slow Alfvén wave (“ion cyclotron wave”) computed by the WKB method. Also shown are the real (RKX) and imaginary (IKX) parts of the wave vector k_x being the solution of local dispersion equation corresponding to the slow mode. Only the mode travelling from the conversion region ($x_c = 175$ cm) towards the minority cyclotron resonance point need to be taken into account here. The result of the comparison of this method with the exact solution is shown in Fig. 2 where real part of E_x component is shown for the new method (SOL), test solution (TEST), as well as for the model solution \mathbf{E}_{mod} (MOD) and the correction field $\delta\mathbf{E}$ (CORR). As one can see, the resonant layer method is in good agreement with the exact test solution.

Note that new method is not limited to differential equations only but is applicable to general problems of integro-differential type. Thus, any kind of nonlocal plasma conductivity can be treated without any further approximations to the conductivity operator. The new method has a straightforward generalization to higher dimensional problems where it should give a significant speed-up of the computations because the maximum number of Fourier harmonics needed is rather small (usually less or of the order of a hundred).

References

[1] A.I.Akhiezer, et al, Plasma Electrodynamics, Pergamon Press, London (1975)

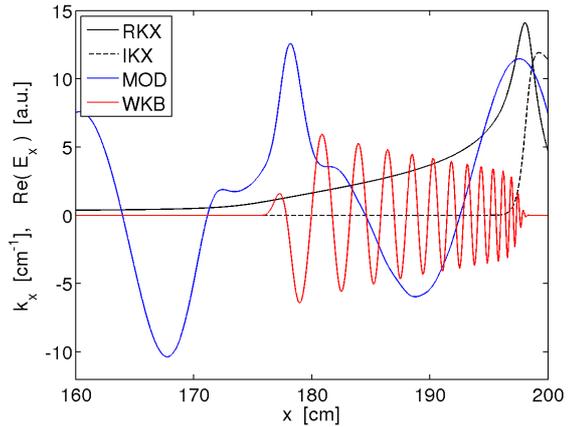


Figure 1: Real part of E_x component. Blue - fast wave model solution, red - WKB model solution.

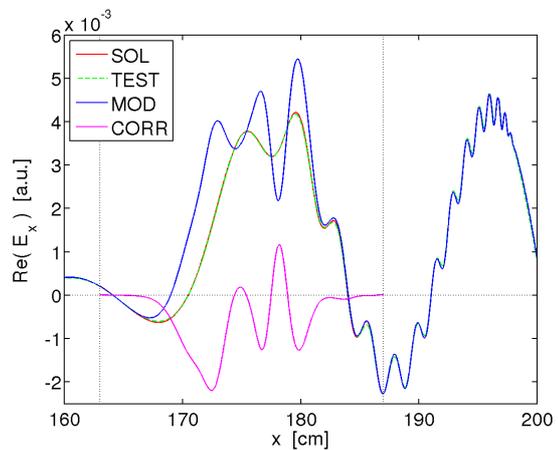


Figure 2: Real part of E_x component. Dotted vertical lines - boundaries of the resonant layer.