A Boundary Integral Formulation for the Dynamic Behavior of a Timoshenko Beam

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Abstract

Mostly, the fourth order differential equation for the deflection is applied in Timoshenko's beam theory although the second order system combining the deflection and the rotation is much more suitable. Considering this system in the Laplace domain, it is straight forward to determine the respective fundamental solutions by Hörmander's method and to obtain the corresponding system of integral equations via the weighted residuum method. Here, avoiding the highly complicated time-dependent fundamental solutions the Convolution Quadrature Method proposed by Lubich is applied resulting in a time stepping formulation for the determination of the time history.
For demonstrating the influence of shear deformation and rotary inertia, examples of a fixed-free and a fixed-simply supported beam are analysed in frequency domain as well as in time domain.

1 Introduction

The importance of shear deformation and rotary inertia in the description of the dynamic response of beams is well documented and an improved theory was given by Timoshenko already in 1921 [1]. An especially important aspect in dynamic analysis of beams is the prediction of natural frequencies and mode shapes. The effect of rotary inertia and shear deformation on the first mode of vibration is small, but it increases rapidly for the second and higher modes [2]. Beside these effects in frequency domain, in time domain Timoshenko's theory has a hyperbolic character opposite to the Euler-Bernoulli theory, resulting in two wave fronts in a transient loaded beam [3].
Since in structural mechanics finite element method is mostly applied in practice, also for Timoshenko beams several studies are known which present different element types and formulations. The Kapur element [4], for example, is based on a cubic displacement expansion for both the bending displacement (that part of the transverse displacement that is attributed to flexural deformation) and the shear displacement (that part due to shear deformation), while other formulations are written in terms of the transverse beam

deflection and normal rotation, derived either from Gurtin's variational principle [5] or from the mixed principles of Reissner and Hu-Washizu [6].

In recent years, the boundary element method (BEM) has been developed for the numerical solution of various engineering mechanic problems as an alternative to the finite element method. For Euler-Bernoulli beams, the BEM is already applied to free and forced flexural vibration problems [7] and to transient analysis [8]. For the refined theory of Timoshenko a detailed paper concerning fundamental solutions and integral representation has been submitted [9].

The governing differential equation for a Timoshenko beam is mostly written in a fourth order equation in deflection with respect to spatial and temporal variable. Here, the more suitable coupled system of two second order differential equations for deflection and rotation is used. In Laplace domain, fundamental solutions for this system are found with the method of Hörmander. A weighted residual formulation gives after two integrations by part a system of integral equations for the deflection and the rotation. To avoid the highly complicated time-dependent fundamental solutions [10], the Convolution Quadrature Method proposed by Lubich [11] is applied resulting in a time stepping formulation for the determination of the time history of the deflection, rotation, bending moment, and shear force.

2 Governing equations and fundamental solutions

Consider a thin beam of length ℓ undergoing transverse motion $w(x,t)$ caused by a load per length $q_z(x,t)$ and, additionally, by a moment per length $m_y(x,t)$. As depicted in Fig. 1, the longitudinal coordinate is x, the flexural rigidity EI_y, the density ρ, the shear modulus G, and the beam cross section A.

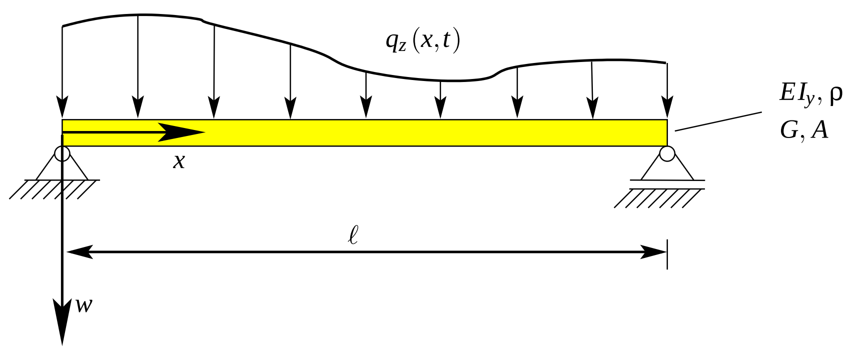
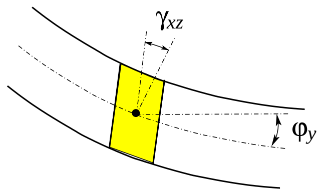


Figure 1: Sketch of the beam under consideration: coordinate system and notation

During vibration, the elements of a beam perform not only a translatory motion but also rotate. Hence, when taking into account not only the rotatory inertia but also the deflection due to shear, the slope of the deflection curve $w(x,t)$ depends not only on the rotation $\varphi_y(x,t)$ of the beam cross section but also on the shear, i.e., on the angle of shear $\gamma_{xz}(x,t)$

at the neutral axis:



$$\frac{\partial}{\partial x} w(x,t) = -\varphi_y(x,t) + \gamma_{xz}(x,t) \tag{1}$$

From the elementary Euler-Bernoulli theory of bending one has the following relations for the bending moment $M_y(x,t)$ and the shear force $Q_z(x,t)$

$$M_y(x,t) = EI_y \frac{\partial}{\partial x} \varphi_y(x,t) \tag{2a}$$

$$Q_z(x,t) = \kappa G A \gamma_{xz}(x,t) = \kappa G A \left(\frac{\partial w(x,t)}{\partial x} + \varphi_y(x,t) \right) \tag{2b}$$

in which κ, the so-called shear coefficient is a factor which gives the ratio of the average shear strain on a section to the shear strain at the centroid (for details, see, e.g., Cowper [12]). Its value is dependent on the shape of cross section, but also, as pointed out by Cowper [12], on the materials Poisson's ratio ν, and, moreover, for dynamic problems on the considered frequency range. Mostly, the values determined for a static analysis $\kappa_{\text{circle}} = 0.9$ and $\kappa_{\text{rectangle}} = 5/6$ for a circular and a rectangular cross section, respectively, are used.
Considering the dynamic equilibrium, i.e., writing the equation of motion in the vertical z-direction and substituting the constitutive relations (2), the following coupled system of differential equations is obtained

$$\kappa G A \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \varphi_y}{\partial x} \right) - \rho A \frac{\partial^2 w}{\partial t^2} = -q_z \tag{3a}$$

$$-\kappa G A \left(\frac{\partial w}{\partial x} + \varphi_y \right) + EI_y \frac{\partial^2 \varphi_y}{\partial x^2} - \rho I_y \frac{\partial^2 \varphi_y}{\partial t^2} = -m_y \,. \tag{3b}$$

Contrary to most other publications, here, these two differential equations of second order are used instead of rearranging these to one differential equation of fourth order.
Applying the Laplace transform to equation (3) and introducing a differential operator notation (i.e., $\partial_x = \partial/\partial x$) leads to the governing set of differential equations for the Timoshenko beam

$$\begin{bmatrix} D_1 \partial_{xx} - S_1 & D_1 \partial_x \\ -D_1 \partial_x & D_2 \partial_{xx} - D_1 - S_2 \end{bmatrix} \begin{bmatrix} \hat{w} \\ \hat{\varphi}_y \end{bmatrix} = - \begin{bmatrix} \hat{q}_z \\ \hat{m}_y \end{bmatrix} \tag{4}$$

with $D_1 = \kappa G A$, $D_2 = EI_y$, $S_1 = \rho A s^2$, and $S_2 = \rho I_y s^2$. Further, in equation (4), the Laplace transform of a function is denoted by $\mathscr{L}\{f(t)\} = \hat{f}(s)$ with the complex variable s.
The so-called fundamental solution matrix $\mathbf{G}$ of the system (4), i.e., the solutions due to a unit impulsive point force $\hat{q}_z^{\infty}(x,\xi) = \delta(x-\xi)$ and to a unit impulsive moment $\hat{m}_y^{\infty}(x,\xi) =$

$\delta(x-\xi)$, respectively, at the point ξ, is defined by $\mathbf{BG}+\mathbf{I}\delta(x-\xi)=\mathbf{0}$, i.e., explicitly

$$\underbrace{\begin{bmatrix} D_1\partial_{xx}-S_1 & D_1\partial_x \\ -D_1\partial_x & D_2\partial_{xx}-D_1-S_2 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \hat{w}_q^\infty(x,\xi,s) & \hat{w}_m^\infty(x,\xi,s) \\ \hat{\varphi}_q^\infty(x,\xi,s) & \hat{\varphi}_m^\infty(x,\xi,s) \end{bmatrix}}_{\mathbf{G}} = -\begin{bmatrix} \delta(x-\xi) & 0 \\ 0 & \delta(x-\xi) \end{bmatrix}. \tag{5}$$

Following the ideas of Hörmander [13], the fundamental solutions are found from a scalar function φ via the ansatz

$$\mathbf{G}=\mathbf{B}^{co}\varphi \tag{6}$$

with the matrix of cofactors $\mathbf{B}^{co}$ of the operator matrix $\mathbf{B}$. Using this ansatz (6) in equation (5) and taking into account the definition of the inverse matrix

$$\mathbf{B}^{-1}=\frac{\mathbf{B}^{co}}{\det(\mathbf{B})} \tag{7}$$

a more convenient form of the governing equation is achieved

$$\mathbf{B}\mathbf{B}^{co}\varphi+\mathbf{I}\delta(\mathbf{x}-\mathbf{y})=\det(\mathbf{B})\,\mathbf{I}\varphi+\mathbf{I}\delta(\mathbf{x}-\mathbf{y})=\mathbf{0}\,. \tag{8}$$

Now, with the two roots of $\det(\mathbf{B})=0$

$$\lambda_{1,2}=\frac{1}{2D_1D_2}\left[(D_1S_2+D_2S_1)\pm\sqrt{(D_1S_2+D_2S_1)^2-4D_1D_2S_1(D_1+S_2)}\right] \tag{9}$$

the problem is reduced to find a solution of the iterated Helmholtz operator

$$(\partial_{xx}-\lambda_1)(\partial_{xx}-\lambda_2)\varphi=-\frac{\delta(x-\xi)}{D_1D_2}\,. \tag{10}$$

This solution is known (see, e.g., [14])

$$\varphi=\frac{1}{2D_1D_2}\frac{1}{\lambda_1-\lambda_2}\left[\frac{e^{-\sqrt{\lambda_1}r}}{\sqrt{\lambda_1}}-\frac{e^{-\sqrt{\lambda_2}r}}{\sqrt{\lambda_2}}\right] \quad \text{with } r=|x-\xi|\,. \tag{11}$$

The remaining step is to apply the operator $\mathbf{B}^{co}$ on the scalar function φ to obtain the fundamental solutions due to a unit force impulse at $x=\xi$ as

$$\hat{w}_q^\infty(x,\xi)=\frac{1}{2D_1}\frac{1}{\lambda_1-\lambda_2}\left[\frac{e^{-\sqrt{\lambda_1}r}}{\sqrt{\lambda_1}}\left(\lambda_1-\frac{D_1+S_2}{D_2}\right)-\frac{e^{-\sqrt{\lambda_2}r}}{\sqrt{\lambda_2}}\left(\lambda_2-\frac{D_1+S_2}{D_2}\right)\right] \tag{12}$$

$$\hat{\varphi}_q^\infty(x,\xi)=\frac{1}{2D_2}\frac{2H(x-\xi)-1}{\lambda_1-\lambda_2}\left[-e^{-\sqrt{\lambda_1}r}+e^{-\sqrt{\lambda_2}r}\right] \tag{13}$$

and due to a unit impulsive moment at $x=\xi$ as

$$\hat{w}_m^\infty(x,\xi)=-\hat{\varphi}_q^\infty(x,\xi) \tag{14}$$

$$\hat{\varphi}_m^{\infty}(x,\xi) = \frac{1}{2D_2}\frac{1}{\lambda_1-\lambda_2}\left[\frac{\mathrm{e}^{-\sqrt{\lambda_1}r}}{\sqrt{\lambda_1}}\left(\lambda_1-\frac{S_1}{D_1}\right)-\frac{\mathrm{e}^{-\sqrt{\lambda_2}r}}{\sqrt{\lambda_2}}\left(\lambda_2-\frac{S_1}{D_1}\right)\right] . \tag{15}$$

In the above $H(x)$ denotes the Heaviside function. Details of the derivation may be found in [9].

3 Boundary integral equation

The most general methodology to derive from differential equations equivalent integral equations is the method of weighted residuals since it is applicable without other specific informations like reciprocal theorems.
For the above equation system (4), the starting equation is its residuum weighted by the matrix $\mathbf{G}$ of the relevant fundamental solutions (12) to (15), i.e., it holds for $\xi \in [0,\ell]$

$$\int_0^{\ell}\left(\mathbf{B}\begin{bmatrix}\hat{w}(x,s)\\ \hat{\varphi}_y(x,s)\end{bmatrix}+\begin{bmatrix}\hat{q}_z(x)\\ \hat{m}_y(x)\end{bmatrix}\right)^T\begin{bmatrix}\hat{w}_q^{\infty}(x,\xi,s) & \hat{w}_m^{\infty}(x,\xi,s)\\ \hat{\varphi}_q^{\infty}(x,\xi,s) & \hat{\varphi}_m^{\infty}(x,\xi,s)\end{bmatrix}\mathrm{d}x = 0 . \tag{16}$$

As usual in boundary integral techniques, the differential operator matrix $\mathbf{B}$ acting on the deflection $\hat{w}(x,s)$ and on the rotation $\hat{\varphi}_y(x,s)$ is shifted by partial integration to act on the matrix of fundamental solutions $\mathbf{G}$. Subsequently, the filtering effect of the Dirac distribution $\delta(x-\xi)$ is taken into account yielding a “boundary” integral equation for $0<\xi<\ell$ (the dependence on s is neglected in the notation)

$$\begin{aligned}\begin{bmatrix}\hat{w}(\xi)\\ \hat{\varphi}_y(\xi)\end{bmatrix} &= \int_0^{\ell}\begin{bmatrix}\hat{w}_q^{\infty}(x,\xi) & \hat{w}_m^{\infty}(x,\xi)\\ \hat{\varphi}_q^{\infty}(x,\xi) & \hat{\varphi}_m^{\infty}(x,\xi)\end{bmatrix}^T\begin{bmatrix}\hat{q}_z(x)\\ \hat{m}_y(x)\end{bmatrix}\mathrm{d}x\\ &+\left[\begin{bmatrix}\hat{w}_q^{\infty}(x,\xi) & \hat{w}_m^{\infty}(x,\xi)\\ \hat{\varphi}_q^{\infty}(x,\xi) & \hat{\varphi}_m^{\infty}(x,\xi)\end{bmatrix}^T\begin{bmatrix}\hat{Q}_z(x)\\ \hat{M}_y(x)\end{bmatrix}-\begin{bmatrix}\hat{Q}_{zq}^{\infty}(x,\xi) & \hat{Q}_{zm}^{\infty}(x,\xi)\\ \hat{M}_{yq}^{\infty}(x,\xi) & \hat{M}_{ym}^{\infty}(x,\xi)\end{bmatrix}^T\begin{bmatrix}\hat{w}(x)\\ \hat{\varphi}_y(x)\end{bmatrix}\right]_{x=0}^{x=\ell}\end{aligned} \tag{17}$$

where the shear forces $\hat{Q}_{zq}^{\infty}(x,\xi)$ and $\hat{Q}_{zm}^{\infty}(x,\xi)$ and the bending moments $\hat{M}_{yq}^{\infty}(x,\xi)$ and $\hat{M}_{ym}^{\infty}(x,\xi)$ are determined by applying equations (2) on the fundamental solutions (12) to (15).
Since always two of the state variables $(\hat{w},\hat{\varphi}_y,\hat{M}_y,\hat{Q}_z)$ at each boundary point, i.e., at $x=0$ and $x=\ell$, are unknown, one needs also two boundary equations at these two boundary points. The simplest way is to evaluate the above system (17) at these two points, i.e., to perform point collocation at $\xi=0+\varepsilon$ and $\xi=\ell-\varepsilon$ with the limit $\varepsilon\to 0$. In this limit, only the two discontinuous functions in (17)

$$\hat{Q}_{zq}^{\infty}(0,0+\varepsilon)=\frac{1-2H(0-0-\varepsilon)}{2}=\frac{1}{2} , \quad \hat{Q}_{zq}^{\infty}(\ell,\ell-\varepsilon)=\frac{1-2H(\ell-\ell+\varepsilon)}{2}=-\frac{1}{2}$$

$$\tag{18}$$

$$\hat{M}^{\infty}_{ym}(0,0+\varepsilon)=\frac{1-2H(0-0-\varepsilon)}{2}=\frac{1}{2}\,,\;\hat{M}^{\infty}_{ym}(\ell,\ell-\varepsilon)=\frac{1-2H(\ell-\ell+\varepsilon)}{2}=-\frac{1}{2} \tag{19}$$

has to be treated separately. With these limits, finally, the "boundary" integral equation for $\xi=0,\ell$ is achieved

$$\begin{aligned}\frac{1}{2}\begin{bmatrix}\hat{w}(\xi)\\ \hat{\varphi}_y(\xi)\end{bmatrix}&=\int_0^\ell\begin{bmatrix}\hat{w}^{\infty}_q(x,\xi) & \hat{w}^{\infty}_m(x,\xi)\\ \hat{\varphi}^{\infty}_q(x,\xi) & \hat{\varphi}^{\infty}_m(x,\xi)\end{bmatrix}^T\begin{bmatrix}\hat{q}_z(x)\\ \hat{m}_y(x)\end{bmatrix}\mathrm{d}x\\ &+\left[\begin{bmatrix}\hat{w}^{\infty}_q(x,\xi) & \hat{w}^{\infty}_m(x,\xi)\\ \hat{\varphi}^{\infty}_q(x,\xi) & \hat{\varphi}^{\infty}_m(x,\xi)\end{bmatrix}^T\begin{bmatrix}\hat{Q}_z(x)\\ \hat{M}_y(x)\end{bmatrix}-\begin{bmatrix}\hat{Q}^{\infty}_{zq}(x,\xi) & \hat{Q}^{\infty}_{zm}(x,\xi)\\ \hat{M}^{\infty}_{yq}(x,\xi) & \hat{M}^{\infty}_{ym}(x,\xi)\end{bmatrix}^T\begin{bmatrix}\hat{w}(x)\\ \hat{\varphi}_y(x)\end{bmatrix}\right]_{x=0}^{x=\ell}.\end{aligned} \tag{20}$$

To obtain a time-dependent integral equation, a formal inverse Laplace transform has to be applied on (17). Every product there is changed to a convolution with respect to time resulting in

$$\begin{aligned}\frac{1}{2}\begin{bmatrix}w(\xi,t)\\ \varphi_y(\xi,t)\end{bmatrix}&=\int_0^t\int_0^\ell\begin{bmatrix}w^{\infty}_q(x,\xi,t-\tau) & w^{\infty}_m(x,\xi,t-\tau)\\ \varphi^{\infty}_q(x,\xi,t-\tau) & \varphi^{\infty}_m(x,\xi,t-\tau)\end{bmatrix}^T\begin{bmatrix}q_z(x,\tau)\\ m_y(x,\tau)\end{bmatrix}\mathrm{d}x\,\mathrm{d}\tau\\ &+\int_0^t\left[\begin{bmatrix}w^{\infty}_q(x,\xi,t-\tau) & w^{\infty}_m(x,\xi,t-\tau)\\ \varphi^{\infty}_q(x,\xi,t-\tau) & \varphi^{\infty}_m(x,\xi,t-\tau)\end{bmatrix}^T\begin{bmatrix}Q_z(x,\tau)\\ M_y(x,\tau)\end{bmatrix}\right.\\ &\left.-\begin{bmatrix}Q^{\infty}_{zq}(x,\xi,t-\tau) & Q^{\infty}_{zm}(x,\xi,t-\tau)\\ M^{\infty}_{yq}(x,\xi,t-\tau) & M^{\infty}_{ym}(x,\xi,t-\tau)\end{bmatrix}^T\begin{bmatrix}w(x,\tau)\\ \varphi_y(x,\tau)\end{bmatrix}\right]_{x=0}^{x=\ell}\mathrm{d}\tau\,.\end{aligned} \tag{21}$$

Hence, time-dependent fundamental solutions are needed. Due to the lack of these solutions, the "Convolution Quadrature Method" proposed by Lubich [11] ($n=0,1,\ldots,N$)

$$y(t)=\int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau\;\Rightarrow y(n\Delta t)=\sum_{k=0}^{n}\omega_{n-k}\left(\hat{f},\Delta t\right)g(k\Delta t)\;, \tag{22}$$

with the integration weights

$$\omega_n\left(\hat{f},\Delta t\right)=\frac{\mathscr{R}^{-n}}{L}\sum_{\ell=0}^{L-1}\hat{f}\left(\frac{\gamma\left(\mathscr{R}\mathrm{e}^{i\ell\frac{2\pi}{L}}\right)}{\Delta t}\right)\mathrm{e}^{-in\ell\frac{2\pi}{L}}, \tag{23}$$

is used. In (22) and (23), the time t is discretized in N time steps of equal duration Δt, and $\gamma(z)$ denotes the quotient of characteristic polynomials of the underlying multi-step method. In the following, a backward differential formula of order two will be used as well as the choice $L=N$.

Applying this quadrature rule on (21), a time-stepping procedure for $n = 0, 1, \ldots, N$ is achieved

$$\begin{aligned} \frac{1}{2}\begin{bmatrix} w(\xi,\Delta t) \\ \varphi_y(\xi,\Delta t)\end{bmatrix} = & \sum_{k=0}^{n}\int_0^{\ell} \omega_{n-k}\left(\begin{bmatrix} \hat{w}_q^{\infty}(x,\xi) & \hat{w}_m^{\infty}(x,\xi) \\ \hat{\varphi}_q^{\infty}(x,\xi) & \hat{\varphi}_m^{\infty}(x,\xi)\end{bmatrix}^T, \Delta t\right)\begin{bmatrix} q_z(x,k\Delta t) \\ m_y(x,k\Delta t)\end{bmatrix}\mathrm{d}x \\ & + \sum_{k=0}^{n}\left[\omega_{n-k}\left(\begin{bmatrix} \hat{w}_q^{\infty}(x,\xi) & \hat{w}_m^{\infty}(x,\xi) \\ \hat{\varphi}_q^{\infty}(x,\xi) & \hat{\varphi}_m^{\infty}(x,\xi)\end{bmatrix}^T, \Delta t\right)\begin{bmatrix} Q_z(x,k\Delta t) \\ M_y(x,k\Delta t)\end{bmatrix}\right. \\ & \left. - \omega_{n-k}\left(\begin{bmatrix} \hat{Q}_{zq}^{\infty}(x,\xi) & \hat{Q}_{zm}^{\infty}(x,\xi) \\ \hat{M}_{yq}^{\infty}(x,\xi) & \hat{M}_{ym}^{\infty}(x,\xi)\end{bmatrix}^T, \Delta t\right)\begin{bmatrix} w(x,k\Delta t) \\ \varphi_y(x,k\Delta t)\end{bmatrix}\right]_{x=0}^{x=\ell} \end{aligned} \tag{24}$$

whereas the discontinuous behavior (18) and (19) can be transfered to the time domain. Note, with the quadrature rule (22) only Laplace transformed fundamental solutions are required to obtain the time stepping procedure (24). Further applications of this technique in various boundary element formulations may be found in [15].

4 Numerical example

In the following, a steel beam with two different support combinations is considered. The material data are given in the Figs. 2 and 5, respectively, and in all examples a frequency independent shear coefficient for a rectangular cross section $\kappa = 5/6$ is used.
First, consider a fixed-free supported beam which is subjected to an impact load $q_z(x,t) = PH(t)\delta(x-\ell)$ with intensity $P = 1000\,\mathrm{N}$ at its free end (see, Fig. 2). By the proposed

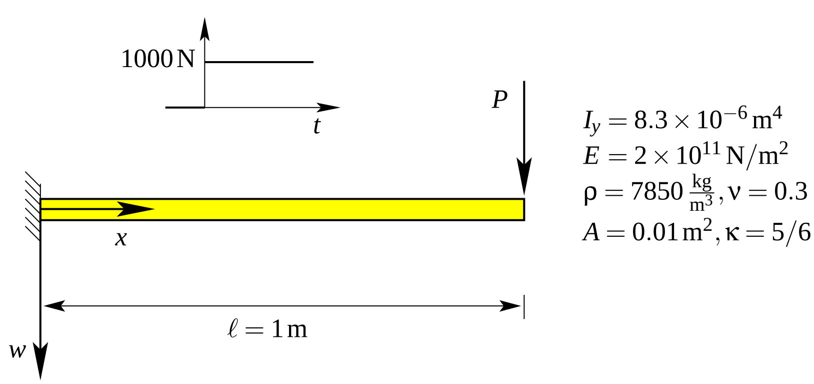


Figure 2: Fixed-free supported beam

method, the deflection, the rotation, the moment, and the shear force can be calculated in frequency ($s = \mathrm{i}\omega$) and time domain. Here, the deflection $\hat{w}(x = \ell, t)$ at the free end is studied.
In Fig. 3, the frequency response of the deflection for a Timoshenko beam as well as for a Euler-Bernoulli beam is plotted in a logarithmic scale up to 30000 Hz. The results using

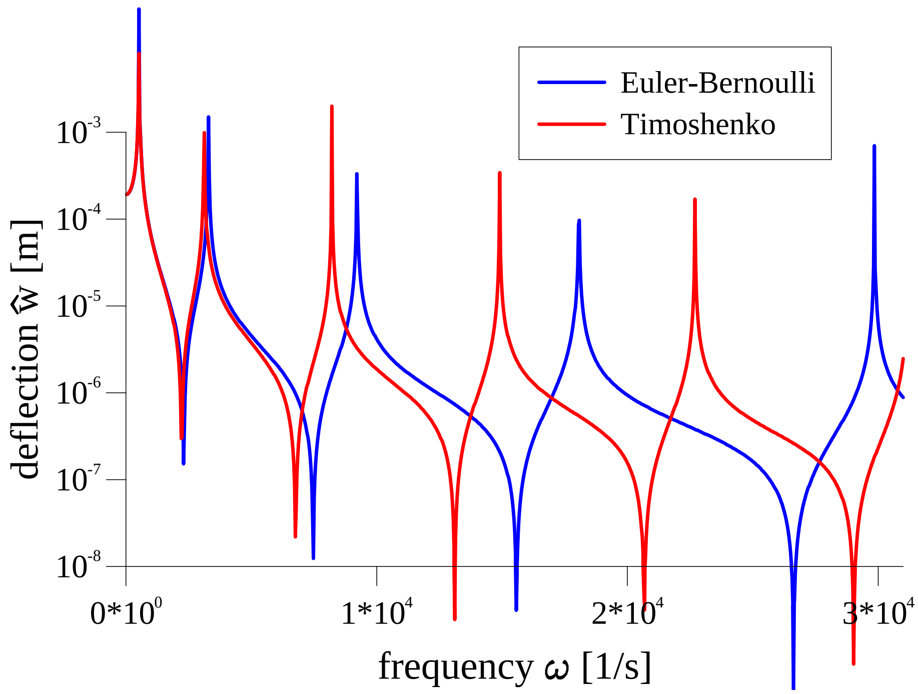


Figure 3: Deflection $\hat{w}$ at the free end versus frequency: Comparison Euler-Bernoulli theory to Timoshenko theory

these two theories are obviously different in higher frequency range. The first eigenfrequency coincides exactly and the second nearly whereas the higher eigenfrequencies of the Timoshenko beam theory are much smaller.
In time domain, the frequency domain effects result in a slight, nearly not visible, phase shift of the time history. In Fig. 4, the time history of the deflection $w(x=\ell,t)$ versus time confirms this. As time step size, for the Timoshenko beam and for the Euler-Bernoulli beam, $\Delta t = 0.00025\,\mathrm{s}$ was chosen.
Next, a fixed-simply supported beam which is subjected to an impact load $q_z(x,t) = PH(t)\,\delta(x-\ell/2)$ with intensity $P = 1000\,\mathrm{N}$ at its midspan is considered. Its geometry and its material data are the same as before (see, Fig. 5). Here, the rotation $\hat{\varphi}_y(x=\ell,\omega)$ at the simply supported end is studied.
In Fig. 6, the frequency response of the rotation for a Timoshenko beam as well as for a Euler-Bernoulli beam is plotted in a logarithmic scale up to 50000 Hz. As before, the results difference between these two theories in higher frequency range is obvious. However, now, only the first eigenfrequency coincides whereas even the second eigenfrequency and, again, the higher eigenfrequencies of the Timoshenko beam are much smaller.
Consequently, in time domain the phase shift of the time history is much more pronounced

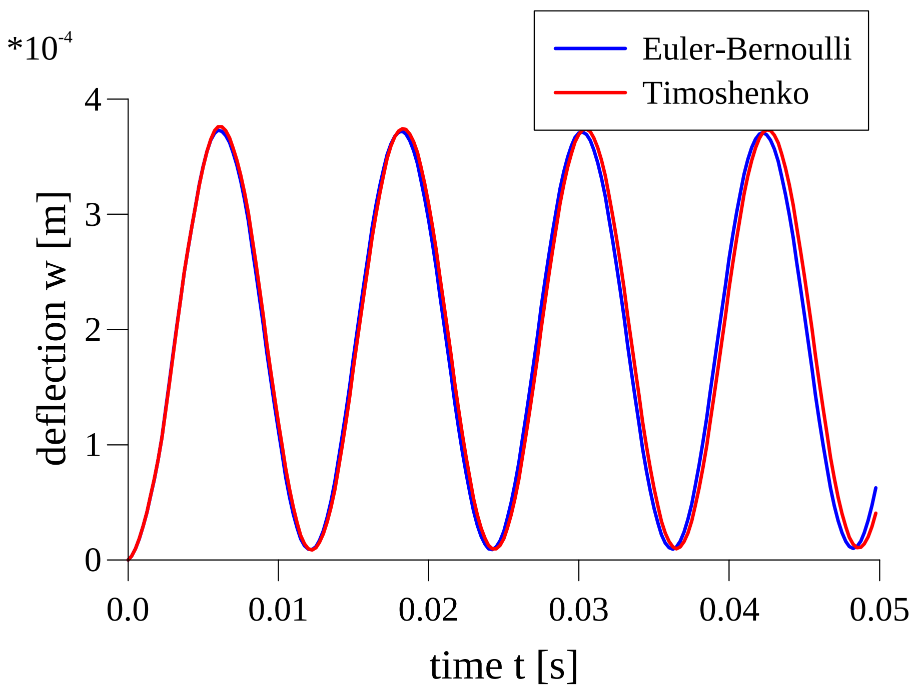


Figure 4: Deflection w at the free end versus time: Comparison Euler-Bernoulli theory to Timoshenko theory

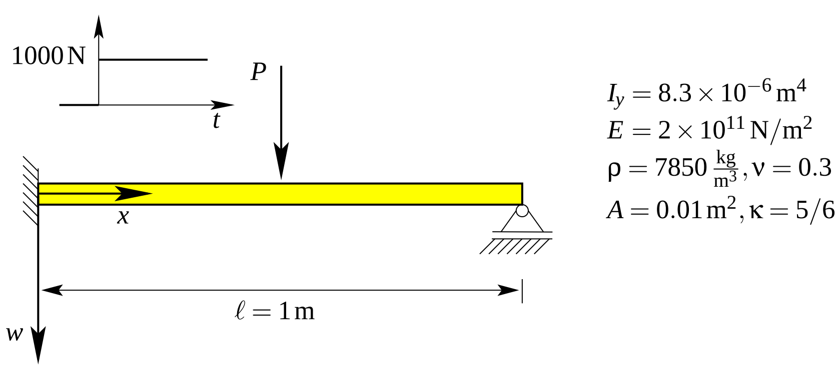


Figure 5: Fixed-simply supported beam

compared to the deflection results of the fixed-free beam. In Fig. 7, the time history of the rotation $\varphi_y(x=\ell,t)$ versus time confirms this. As time step size, for the Timoshenko beam $\Delta t = 0.0002\,\mathrm{s}$ and for the Euler-Bernoulli beam $\Delta t = 0.0001\,\mathrm{s}$ was chosen to obtain the results with the smallest difference in amplitude. However, there is still a difference

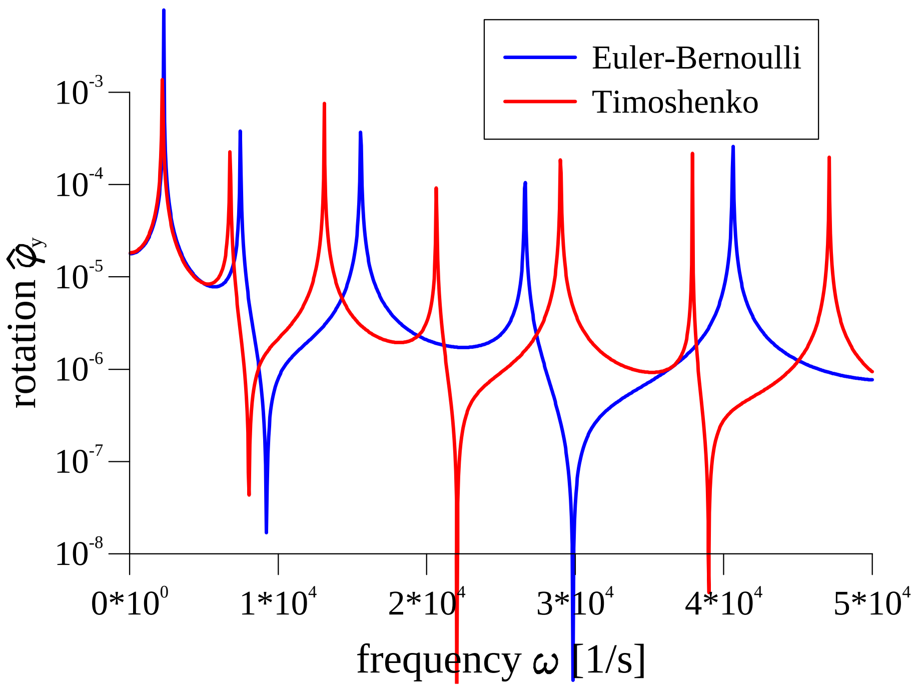


Figure 6: Rotation $\hat{\varphi}_y$ at the simply supported end versus frequency: Comparison Euler-Bernoulli theory to Timoshenko theory

in amplitude which is presumably caused by the different numerical behavior of both theories, whereas the phase shift is not influenced by the time step size. In general, it can be stated from numerical experience that the Timoshenko theory yields a more stable numerical algorithm. This is presumably caused by the hyperbolic character of this theory while the Euler-Bernoulli theory is parabolic.

5 Conclusions

In the paper at hand, a boundary integral formulation for a Timoshenko beam, i.e., the fundamental solution as well as the integral equation is presented. The time response is achieved by applying the 'Convolution Quadrature Method' of Lubich. The examples of a fixed-free and of a fixed-simply supported beam show the accuracy of the proposed method as well as the difference of responses achieved with an Euler-Bernoulli theory

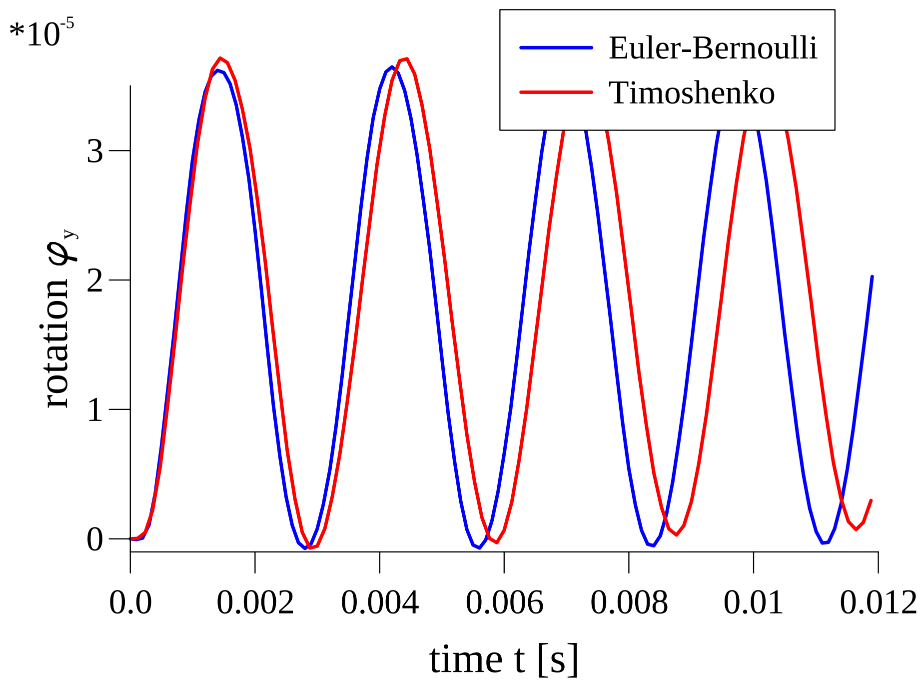


Figure 7: Rotation φ_y at the simply supported end versus time: Comparison Euler-Bernoulli theory to Timoshenko theory

compared to those of the Timoshenko theory. Only the first eigenfrequency coincides whereas the higher eigenfrequencies of the Timoshenko beam are smaller than that of the Euler-Bernoulli beam. In time domain, clearly, a phase shift between both theories is observed.

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