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# Trimmed stable AR(1) processes

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## Abstract

In this paper we investigate the distribution of trimmed sums of dependent observations with heavy tails. We consider the case of autoregressive processes of order one with independent innovations in the domain of attraction of a stable law. We show if the  $d$  largest (in magnitude) terms are removed from the sample, then the sum of the remaining elements satisfies a functional central limit theorem with random centering provided  $d = d(n) \geq n^\gamma$  (for some  $\gamma > 0$ ) and  $d(n)/n \rightarrow 0$ . This result is used to get asymptotics for the widely used CUSUM process in case of dependent heavy tailed observations.

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## 1. Introduction

Let  $X_1, X_2, \dots$ , be independent, identically distributed random variables in the domain of attraction of a stable law with index  $0 < \alpha < 2$ . Lévy [38] and Darling [22] noted that the order of magnitude of the sum  $S_n = \sum_{k=1}^n X_k$  is the same as that of its largest term and the contribution of a fixed, but large number of extremal terms is essentially responsible for the distribution of  $S_n$ . The asymptotic distribution of the trimmed sum  $S_n^{(d)}$  obtained from  $S_n$  by discarding the  $d$  smallest and  $d$  largest summands was determined by Lepage et al. [37] and Csörgő et al. [21]

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proved that in case of moderate trimming, i.e.  $d(n) \rightarrow \infty, d(n)/n \rightarrow 0$  the trimmed sum  $S_n^{(d)}$  satisfies the central limit theorem. Heavy trimming, i.e. when  $0 < \lim_{n \rightarrow \infty} d(n)/n < 1$ , is studied in [17–19]. Arov and Bobrov [2], Mori [43], Hall [33], Teugels [48], Griffin and Pruitt [31,32] and Kesten [35] considered a different type of trimming of the sample. Let  $\eta_{n,d}$  denote the  $d$ th largest element of  $|X_1|, \dots, |X_n|$ . These authors were interested in the asymptotic behavior of the modulus trimmed sum  ${}^{(d)}S_n = \sum_{k=1}^n X_k I\{|X_k| \leq \eta_{n,d}\}$ , i.e. when from the sum we remove the  $d$  elements with the largest absolute values. Griffin and Pruitt [31] proved that the trimmed central limit theorem of Csörgő et al. [21] remains valid for modulus trimmed sums provided the distribution of  $X_1$  is symmetric, but it generally fails for nonsymmetric variables and it can happen that  ${}^{(d)}S_n$  is asymptotically normal for some  $d(n)$ , but not for another  $d'(n) \geq d(n)$ . This is somewhat unexpected, since removing more large elements from the sample should result in better behavior. Sufficient conditions for the asymptotic normality of  ${}^{(d)}S_n$  in the nonsymmetric case were given in [9]. On the other hand, Berkes et al. [11] showed that if  $d(n) \rightarrow \infty, d(n)/n \rightarrow 0$ , a functional central limit theorem always holds for  ${}^{(d)}S_n$  with a random centering factor. Some of these results are extended in [36] to long range dependent sequences.

Trimming also has important applications in statistics. As an example, we consider the detection of possible changes in the location model

$$X_j = c_j + e_j, \quad 1 \leq j \leq n,$$

where  $e_1, \dots, e_n$  are random errors. Under the null hypothesis

$$H_0 : c_1 = c_2 = \dots = c_n$$

of no change in the location parameter we have

$$X_j = c + e_j, \quad 1 \leq j \leq n, \tag{1.1}$$

with some constant  $c$ . Under the alternative there are  $r$  changes:

$$\begin{aligned} H_A : \quad & \text{there is } r \geq 1 \text{ and } 1 < k_1 < k_2 < \dots < k_r < n \text{ such that} \\ & c_1 = \dots = c_{k_1-1} \neq c_{k_1} = c_{k_1+1} = \dots = c_{k_2-1} \neq c_{k_2} = c_{k_2+1} = \dots \\ & = c_{k_r-1} \neq c_{k_r} = \dots = c_n. \end{aligned}$$

The most popular methods to test  $H_0$  against  $H_A$  (cf. [4,20]) are based on the CUSUM process

$$U_n(x) = \sum_{i=1}^{\lfloor nx \rfloor} X_i - \frac{\lfloor nx \rfloor}{n} \sum_{i=1}^n X_i, \tag{1.2}$$

where  $\lfloor \cdot \rfloor$  denotes the integer part. Clearly, if  $H_0$  is true, then  $U_n(t)$  does not depend on the common but unknown location parameter  $c_1$ . It is well known if  $X_1, \dots, X_n$  are independent and identically distributed random variables with a finite second moment, then

$$\frac{1}{(\text{nvar}(X_1))^{1/2}} U_n(x) \xrightarrow{\mathcal{D}[0,1]} B(x),$$

where  $B(x)$  is a Brownian bridge and  $\xrightarrow{\mathcal{D}[0,1]}$  means weak convergence in the space  $\mathcal{D}[0,1]$  of càdlàg functions equipped with the Skorokhod  $J_1$  topology (cf. [12]). Assuming that  $X_1, X_2, \dots, X_n$  are independent and identically distributed random variables in the domain of

attraction of a stable law of index  $\alpha \in (0, 2)$ , Aue et al. [3] showed that

$$\frac{1}{n^{1/\alpha} \hat{L}(n)} U_n(x) \xrightarrow{\mathcal{D}[0,1]} B_\alpha(x),$$

where  $\hat{L}$  is a slowly varying function at  $\infty$  and  $B_\alpha(x)$  is an  $\alpha$ -stable bridge. (The  $\alpha$ -stable bridge is defined as  $B_\alpha(x) = W_\alpha(x) - xW_\alpha(1)$ , where  $W_\alpha$  is a Lévy  $\alpha$ -stable motion.) Since nothing is known on the distributions of the functionals of  $\alpha$ -stable bridges, Berkes et al. [11] suggested the trimmed CUSUM process

$$T_{n,d}(x) = \sum_{i=1}^{\lfloor nx \rfloor} X_i I\{|X_i| \leq \eta_{n,d}\} - \frac{\lfloor nx \rfloor}{n} \sum_{i=1}^n X_i I\{|X_i| \leq \eta_{n,d}\}. \quad (1.3)$$

Assuming that the  $X_i$ 's are independent and identically distributed and are in the domain of attraction of a stable law, they proved

$$\frac{1}{\sigma_n} T_{n,d}(x) \xrightarrow{\mathcal{D}[0,1]} B(x), \quad (1.4)$$

where

$$\sigma_n^2 = \frac{\alpha}{2-\alpha} (H^{-1}(d/n))^2 d,$$

$B(t)$  is a Brownian bridge and  $H^{-1}$  denotes the generalized inverse of  $H$ , the survival function of  $X_1$ . The CUSUM process has also been widely used in case of dependent variables but it is nearly always assumed that the observations have high moments and the dependence in the sequence is weak. For a review we refer to [4]. However, very few papers consider the instability of time series models with heavy tails.

Fama [28] and Mandelbrot [40,41] pointed out that the distributions of commodity and stock returns are often heavy tailed with possible infinite variance and their research started the investigation of time series models where the marginal distributions have regularly varying tails. Davis and Resnick [24,25] investigated the properties of moving averages with regularly varying tails and obtained non-Gaussian limits for the sample covariances and correlations. Their results were extended to heavy tailed ARCH in [23]. The empirical periodogram was studied by Mikosch et al. [42]. Andrews et al. [1] estimated the parameters of autoregressive processes with stable innovations.

In this paper we extend the results of [11] to a dependent setting. More precisely, we prove a functional central limit theorem for trimmed sums with random centering for an AR(1) process with innovations in the domain of attraction of a stable law. As a consequence we derive a limit result similar to (1.4) for the trimmed AR(1) process.

In Section 2 we formulate our main results. The proofs are based on several technical lemmas, given in Section 3. In Section 4 we prove the main results in the case when trimming is replaced by a (nonrandom) truncation and from these in Section 5 we deduce Theorems 2.1 and 2.2.

## 2. Main results

Let  $e_i$  be a non-anticipative (i.e.  $\sigma\{\varepsilon_j, j \leq i\}$  measurable) solution of

$$e_i = \rho e_{i-1} + \varepsilon_i \quad -\infty < i < \infty. \quad (2.1)$$

We assume throughout this paper that

$$\varepsilon_j, -\infty < j < \infty \text{ are independent and identically distributed,} \quad (2.2)$$

$\varepsilon_0$  belongs to the domain of attraction of a stable

$$\text{random variable } \xi^{(\alpha)} \text{ with parameter } 0 < \alpha < 2, \quad (2.3)$$

and

$$\varepsilon_0 \text{ is symmetric when } \alpha = 1. \quad (2.4)$$

Assumption (2.3) means that

$$\left( \sum_{j=1}^n \varepsilon_j - a_n \right) / b_n \xrightarrow{\mathcal{D}} \xi^{(\alpha)} \quad (2.5)$$

for some numerical sequences  $a_n$  and  $b_n$ . The necessary and sufficient condition for this is

$$\lim_{t \rightarrow \infty} \frac{P\{\varepsilon_0 > t\}}{L_*(t)t^{-\alpha}} = p \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{P\{\varepsilon_0 \leq -t\}}{L_*(t)t^{-\alpha}} = q \quad (2.6)$$

for some numbers  $p \geq 0, q \geq 0, p + q = 1$ , where  $L_*$  is a slowly varying function at  $\infty$ . It is known that (2.1) has a unique stationary non-anticipative solution if and only if

$$-1 < \rho < 1. \quad (2.7)$$

Under assumptions (2.2)–(2.7),  $\{e_j\}$  is a stationary sequence and  $E|e_0|^\kappa < \infty$  for all  $0 < \kappa < \alpha$  but  $E|e_0|^\kappa = \infty$  for all  $\kappa > \alpha$ . AR(1) processes with stable innovations were considered by Chan and Tran [15], Chan [14], Aue and Horváth [5] and Zhang and Chan [51] who investigated the case when  $\rho$  is close to 1 and provided estimates for  $\rho$  and the other parameters when the observations do not have finite variances.

The convergence of the finite dimensional distributions of  $U_n(x)$  in the AR(1) case is an immediate consequence of Phillips and Solo [45] representation. Let  $\xrightarrow{fdd}$  denote the convergence of the finite dimensional distributions. If (1.1)–(2.4) and (2.7) hold, then we have

$$\frac{1 - \rho}{n^{1/\alpha} L_*(n)} U_n(x) \xrightarrow{fdd} B_\alpha(x), \quad (2.8)$$

where  $B_\alpha(x), 0 \leq x \leq 1$  is an  $\alpha$ -stable bridge and  $L_*$  is defined in (2.6). It has been pointed out in [6,7] that the  $fdd$  convergence in (2.8) cannot be replaced with weak convergence in  $\mathcal{D}[0, 1]$ . However, Avram and Taqqu [7] proved that  $U_n(x)$  converges in the weak- $M_1$  sense under some additional regularity conditions. Some of their regularity conditions were removed by Tyranskińska [49]. For further results on the weak convergence of dependent sequences with infinite variance in the  $M_1$  topology we refer to [8].

We formulate now our main results. On the truncation parameter  $d = d(n)$  we will assume

$$\lim_{n \rightarrow \infty} d(n)/n = 0 \quad (2.9)$$

and

$$d(n) \geq n^\delta \quad \text{with some } 0 < \delta < 1. \quad (2.10)$$

Let  $F(x) = P\{X_0 \leq x\}$ ,  $H(x) = P\{|X_0| > x\}$  and let  $H^{-1}(t)$  be the (generalized) inverse of  $H$ . Our last condition will be used to establish the weak law of large numbers for  $\eta_{n,d}$ . We assume that  $\varepsilon_0$  has a density function  $p(t)$  which satisfies

$$\int_{-\infty}^{\infty} |p(t+s) - p(t)| dt \leq C|s| \quad \text{with some positive constant } C. \quad (2.11)$$

(Here, and in the sequel, all constants will be finite and positive.) Let

$$A_n = d^{1/2} H^{-1}(d/n) \quad (2.12)$$

and

$$m(t) = EX_1 I\{|X_1| \leq t\}. \quad (2.13)$$

**Theorem 2.1.** *If (1.1), (2.1)–(2.4), (2.7) and (2.9)–(2.11) hold, then we have that*

$$\left(\frac{2-\alpha}{\alpha}\right)^{1/2} \left(\frac{1-\rho}{1+\rho}\right)^{1/2} \frac{1}{A_n} \sum_{k=1}^n [X_k I\{|X_k| \leq \eta_{n,d}\} - m(\eta_{n,d})] \xrightarrow{\mathcal{D}[0,1]} W(x),$$

where  $W(x)$  is a Wiener process.

The result in **Theorem 2.1** uses the random centering factor  $m(\eta_{n,d})$ . This factor is characteristic for the asymptotic distribution of the modulus trimmed partial sums process, as first observed in [11]. Since a random translation of the terms in the CUSUM process cancels out, the next result is an immediate consequence of **Theorem 2.1**.

**Theorem 2.2.** *If (1.1), (2.1)–(2.4), (2.7) and (2.9)–(2.11) hold, then we have that*

$$\left(\frac{2-\alpha}{\alpha}\right)^{1/2} \left(\frac{1-\rho}{1+\rho}\right)^{1/2} \frac{T_{n,d}(x)}{A_n} \xrightarrow{\mathcal{D}[0,1]} B(x),$$

where  $B(x)$  is a Brownian bridge.

Statistical applications of **Theorem 2.2** require the estimation of the norming factor from the observations. We suggest a kernel type estimator for the norming factor in **Theorem 2.2** which is computed from the trimmed observations  $X_i^* = X_i I\{|X_i| \leq \eta_{n,d}\}$ . Let

$$\hat{s}_n^2 = \hat{\gamma}_0 + 2 \sum_{j=1}^{n-1} \omega(j/h) \hat{\gamma}_j,$$

where

$$\hat{\gamma}_j = \frac{1}{n} \sum_{i=1}^{n-j} (X_i^* - \bar{X}_n^*)(X_{i+j}^* - \bar{X}_n^*) \quad \text{with } \bar{X}_n^* = \frac{1}{n} \sum_{i=1}^n X_i^*.$$

For the kernel  $\omega(t)$  we assume the following regularity conditions: (i)  $\omega(0) = 1$ , (ii)  $\omega(t) = 0$  if  $t > a$  with some  $a > 0$ , (iii)  $\omega$  is Lipschitz continuous, and (iv) the Fourier transform of  $\omega$  is Lipschitz continuous and integrable on the real line. Assuming that  $h = h(n) \rightarrow \infty$  and  $h(n)/n \rightarrow 0$ , the method in [34,39] can be used to establish that

$$\frac{\hat{s}_n}{A_n} \xrightarrow{P} \left(\frac{\alpha}{2-\alpha}\right)^{1/2} \left(\frac{1+\rho}{1-\rho}\right)^{1/2}.$$

Hence Theorem 2.2 yields

$$\frac{T_{n,d}(x)}{\hat{s}_n} \xrightarrow{\mathcal{D}[0,1]} B(x),$$

where  $B(x)$  is a Brownian bridge.

In this paper we considered a stationary AR(1) sequence with stable innovations. Theorems 2.1 and 2.2 could be extended to linear processes but this extension would require nontrivial modifications of our method or a completely different approach.

### 3. Preliminary results

The proofs of Theorems 2.1 and 2.2 are based on several technical lemmas.

In the sequel we can and will assume without loss of generality that

$$E\varepsilon_0 = 0, \quad \text{if } 1 < \alpha < 2. \quad (3.1)$$

Under these conditions, in (2.5) we can choose  $a_n = 0$  and  $b_n$  can be chosen any sequence satisfying

$$\frac{n}{b_n^\alpha} L_*(b_n) \rightarrow 1. \quad (3.2)$$

According to Theorem 2.3 of Cline [16] (cf. also Davis and Resnick [25]),  $H(x)$ , the survival function of  $|X_0|$  satisfies

$$H(x) = x^{-\alpha} L(x), \quad (3.3)$$

where  $L(x)$  is a slowly varying function at  $\infty$  and

$$\lim_{x \rightarrow \infty} \frac{H(x)}{P\{|\varepsilon_0| > x\}} = \lim_{x \rightarrow \infty} \frac{L(x)}{L_*(x)} = \frac{1}{1 - |\rho|^\alpha}. \quad (3.4)$$

Let

$$u_{k,n}(t) = X_k I\{|X_k| \leq tH^{-1}(d/n)\} \quad \text{and} \quad m_n(t) = E[X_0 I\{|X_0| \leq tH^{-1}(d/n)\}].$$

The main goal of this section is to get bounds for  $Eu_{0,n}(t)u_{k,n}(s)$  and  $\text{cov}(u_{0,n}(t), u_{k,n}(s))$ .

**Lemma 3.1.** *We assume that (1.1), (2.1)–(2.4), (2.7) and (3.1) hold. Let  $\mathbf{Y}^{(k)} = (X_0, X_k)$  and let  $\mathbf{Y}_i^{(k)}, i = 1, 2, \dots$  be independent and identically distributed copies of  $\mathbf{Y}^{(k)}$ . Then*

$$\frac{\mathbf{Y}_1^{(k)} + \dots + \mathbf{Y}_n^{(k)}}{n^{1/\alpha} L_*(n)} \xrightarrow{\mathcal{D}} \mathbf{Z}^{(k)} \quad \text{as } n \rightarrow \infty,$$

where  $\mathbf{Z}^{(k)} = (Z_1^{(k)}, Z_2^{(k)})$  with

$$Z_1^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \xi_{-\ell}^{(\alpha)} \quad \text{and} \quad Z_2^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \xi_{k-\ell}^{(\alpha)}$$

and  $\xi_\ell^{(\alpha)}, -\infty < \ell < \infty$  are independent and identically distributed copies of  $\xi^{(\alpha)}$ .

**Proof.** It follows from (2.1) that

$$X_k - c = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{k-\ell} = \sum_{\ell=0}^{k-1} \rho^\ell \varepsilon_{k-\ell} + \rho^k X_0, \quad 1 \leq k < \infty. \tag{3.5}$$

Let  $\varepsilon_\ell^{(i)}$ ,  $-\infty < \ell < \infty$ ,  $i = 1, 2, \dots$  be independent and identically distributed copies of  $\varepsilon_0$ . Clearly

$$\mathbf{Y}_i^{(k)} = (Y_{i,1}^{(k)}, Y_{i,2}^{(k)}) \quad \text{with} \quad Y_{i,1}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{-\ell}^{(i)} \quad \text{and} \quad Y_{i,2}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{k-\ell}^{(i)}$$

are independent and identically distributed copies of  $\mathbf{Y}^{(k)}$ . Elementary algebra gives

$$\sum_{i=1}^n Y_{i,1}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \sum_{i=1}^n \varepsilon_{-\ell}^{(i)} \quad \text{and} \quad \sum_{i=1}^n Y_{i,2}^{(k)} = \sum_{\ell=0}^{k-1} \rho^\ell \sum_{i=1}^n \varepsilon_{k-\ell}^{(i)} + \rho^k \sum_{\ell=0}^{\infty} \rho^\ell \sum_{i=1}^n \varepsilon_{-\ell}^{(i)}.$$

For every  $L \geq 0$  by (2.5) we have that (recall that under our conditions the centering factors  $a_n$  in (2.5) can be chosen 0)

$$\frac{1}{b_n} \left( \sum_{i=1}^n \varepsilon_\ell^{(i)}, -L \leq \ell \leq L \right) \xrightarrow{\mathcal{D}} \left( \xi_\ell^{(\alpha)}, -L \leq \ell \leq L \right),$$

where  $\xi_\ell^{(\alpha)}$ ,  $-\infty < \ell < \infty$  are independent and identically distributed copies of  $\xi^{(\alpha)}$ . Let  $0 < \kappa < \alpha$ . It follows from Theorem 6.1 of de Acosta and Giné [27, p. 225] that

$$E \left| \frac{1}{b_n} \sum_{i=1}^n \varepsilon_\ell^{(i)} \right|^\kappa \leq C_1,$$

and therefore for every  $x > 0$  we have that

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \sum_{\ell=L+1}^{\infty} \rho^\ell \left| \frac{1}{b_n} \sum_{i=1}^n \varepsilon_\ell^{(i)} \right| > x \right\} = 0$$

and similarly

$$\lim_{L \rightarrow \infty} P \left\{ \sum_{\ell=L+1}^{\infty} \rho^\ell |\xi_\ell^{(\alpha)}| > x \right\} = 0.$$

This completes the proof of the lemma.  $\square$

Let  $\mathbf{i}$  denote the imaginary unit.

**Lemma 3.2.** Let  $\mathbf{Y}$  be a stable vector variable with characteristic function  $\psi(s, t)$ . Then there exists a measure  $\nu$  on the Borel sets of  $\mathbb{R}^2$  such that for some  $C_1, C_2$  and any  $\gamma > 0$

$$\begin{aligned} \psi(s, t) = \exp \left\{ \mathbf{i}(C_1 s + C_2 t) + \int_{|\mathbf{u}| > \gamma} (e^{\mathbf{i}(s u_1 + t u_2)} - 1) \nu(d u_1, d u_2) \right. \\ \left. + \int_{0 < |\mathbf{u}| \leq \gamma} (e^{\mathbf{i}(s u_1 + t u_2)} - 1 - \mathbf{i}(s u_1 + t u_2)) \nu(d u_1, d u_2) \right\}, \end{aligned}$$

where  $\mathbf{u} = (u_1, u_2)$ .

The result can be found, for example, in Gikhman and Skorohod [29, Chapter 5].  $\nu$  is called the Lévy measure in the canonical representation of the characteristic function of  $\mathbf{Y}$ . The stable vectors in our paper will be centered, i.e.  $C_1 = C_2 = 0$ .

**Lemma 3.3.** *If (1.1), (2.1)–(2.4), (2.7) and (3.1) hold, then we have*

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{T^{\alpha-2}}{L_*(T)} E X_0 I\{|X_0| \leq vT\} X_k I\{|X_k| \leq wT\} \\ = \frac{\alpha}{2-\alpha} \frac{\rho^k}{1-|\rho|^\alpha} (\min(v, w|\rho|^{-k}))^{2-\alpha}. \end{aligned}$$

**Proof.** It follows from Theorem 4 of Resnick and Greenwood [46] that

$$\lim_{n \rightarrow \infty} nP \left\{ \frac{(X_0, X_k)}{b_n} \in \mathbf{A} \right\} = \nu(\mathbf{A}), \quad (3.6)$$

where  $b_n$  is defined in (3.2) and  $\mathbf{A}$  is any Borel set of  $R^2$ , not containing  $(0, 0)$ ,  $\nu(\mathbf{A}) < \infty$  and the  $\nu$ -measure of the boundary of  $\mathbf{A}$  is 0. Since  $nL_*(b_n)/b_n^\alpha \rightarrow 1$ , with the choice of  $n = \lfloor T^\alpha/L_*(T) \rfloor$  we get from (3.6) that

$$\lim_{T \rightarrow \infty} \frac{T^\alpha}{L_*(T)} P\{(X_0, X_k)/T \in \mathbf{A}\} = \nu(\mathbf{A}), \quad (3.7)$$

where  $\nu$  is the Lévy measure in the canonical representation of the characteristic function of  $\mathbf{Z}^{(k)}$ . Denoting the joint distribution of  $X_0/T$  and  $X_k/T$  by  $\nu_k^{(T)}$ , relation (3.7) says

$$\lim_{T \rightarrow \infty} \frac{T^\alpha}{L_*(T)} \mu_k^{(T)}(\mathbf{A}) = \nu(\mathbf{A})$$

for any Borel-set  $\mathbf{A} \subset R^2$  not containing  $(0, 0)$  and having Lebesgue measure 0 for its boundary. Since the function  $f(x, y) = xy$  equals 0 at the origin, using the weak convergence of  $T^\alpha \mu_k^{(T)}/L_*(T)$  over  $D([-v, v] \times [-w, w] \setminus [-s, s] \times [-s, s])$  and letting  $s \downarrow 0$  we get

$$\lim_{T \rightarrow \infty} \frac{T^\alpha}{L_*(T)} \int_{-v}^v \int_{-w}^w xy \nu_k^{(T)}(dx, dy) = \int_{-v}^v \int_{-w}^w xy \nu(dx, dy)$$

which can be written equivalently as

$$\lim_{T \rightarrow \infty} \frac{T^{\alpha-2}}{L_*(T)} E X_0 I\{|X_0| \leq vT\} X_k I\{|X_k| \leq wT\} = \int_{-v}^v \int_{-w}^w xy \nu(dx, dy).$$

Since  $\xi^{(\alpha)}$  is a stable random variable, its characteristic function can be written as  $\exp(-\psi(t))$  and with this notation we get

$$E \exp(i(sZ_1^{(k)} + tZ_2^{(k)})) = \exp \left( - \sum_{\ell=0}^{\infty} \psi(s\rho^\ell + t\rho^{k+\ell}) - \sum_{\ell=0}^{k-1} \psi(t\rho^\ell) \right).$$

If  $\hat{\nu}_\ell$  denotes the Lévy measure associated with the characteristic function  $\exp(-\psi(s\rho^\ell + t\rho^{k+\ell}))$  and  $\tilde{\nu}_\ell$  corresponds to  $\exp(-\psi(t\rho^\ell))$ , then we have

$$\nu(\mathbf{A}) = \sum_{\ell=0}^{\infty} \hat{\nu}_\ell(\mathbf{A}) + \sum_{\ell=0}^{k-1} \tilde{\nu}_\ell(\mathbf{A}).$$



Hence

$$\int_{-v}^v \int_{-w}^w xy v(dx, dy) = \sum_{\ell=0}^{\infty} \int_{-v}^v \int_{-w}^w xy \hat{v}_{\ell}(dx, dy).$$

Next we note that there is a positive constant  $A^*$  such that

$$\lim_{x \rightarrow \infty} \frac{P\{|\xi^{(\alpha)}| > x\}}{x^{-\alpha}} = A^*$$

and therefore by Bingham et al. [13, p. 346] we obtain that

$$\lim_{x \rightarrow \infty} \frac{E(\xi^{(\alpha)})^2 I\{|\xi^{(\alpha)}| \leq x\}}{x^2 P\{|\xi^{(\alpha)}| > x\}} = \frac{\alpha}{2 - \alpha}$$

resulting in

$$\lim_{x \rightarrow \infty} \frac{E(\xi^{(\alpha)})^2 I\{|\xi^{(\alpha)}| \leq x\}}{x^{2-\alpha}} = A^* \frac{\alpha}{2 - \alpha}.$$

The last relation implies

$$\begin{aligned} & \lim_{T \rightarrow \infty} T^{\alpha-2} E \left[ \rho^{2\ell+k} (\xi^{(\alpha)})^2 I\{|\xi^{(\alpha)}| \leq T \min(v|\rho|^{-\ell}, w|\rho|^{-(\ell+k)})\} \right] \\ & = A^* \frac{\alpha}{2 - \alpha} \rho^{2\ell+k} (\min(v|\rho|^{-\ell}, w|\rho|^{-(\ell+k)}))^{2-\alpha}. \end{aligned}$$

We note that  $\exp(-\psi(s\rho^{\ell} + t\rho^{k+\ell}))$  is the characteristic function of the vector  $(\rho^{\ell}\xi^{(\alpha)}, \rho^{k+\ell}\xi^{(\alpha)})$ , so repeating the arguments leading to (3.6) and (3.7) for this vector instead of  $(X_0, X_k)$  we get

$$\begin{aligned} & \lim_{T \rightarrow \infty} \rho^{k+2\ell} \frac{T^{\alpha-2}}{A^*} E \xi^{(\alpha)} I\{|\rho^{\ell}\xi^{(\alpha)}| \leq vT\} \xi^{(\alpha)} I\{|\rho^{k+\ell}\xi^{(\alpha)}| \leq wT\} \\ & = \int_{-v}^v \int_{-w}^w xy \hat{v}_{\ell}(dx, dy), \end{aligned}$$

and therefore

$$\int_{-v}^v \int_{-w}^w xy \hat{v}_{\ell}(dx, dy) = \frac{\alpha}{2 - \alpha} \rho^k |\rho|^{\alpha\ell} (\min(v, w|\rho|^{-k}))^{2-\alpha}.$$

Summing for  $\ell = 0, 1, \dots$ , we get Lemma 3.3.  $\square$

**Lemma 3.4.** *If (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold, then for every  $k = 0, 1, 2, \dots$*

$$\lim_{n \rightarrow \infty} \frac{n E(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))}{A_n^2} = \frac{\alpha}{2 - \alpha} \rho^k (\min(s, t|\rho|^{-k}))^{2-\alpha}. \quad (3.8)$$

**Proof.** If  $1 < \alpha < 2$ , then

$$\lim_{n \rightarrow \infty} m_n(t) = EX_0 \quad \text{for any } t > 0. \quad (3.9)$$

If  $0 < \alpha < 1$ , then

$$|m_n(t)| \leq \int_{-tH^{-1}(d/n)}^{tH^{-1}(d/n)} |x| dF(x)$$

$$= - \int_0^{tH^{-1}(d/n)} x dH(x) = -xH(x) \Big|_0^{tH^{-1}(d/n)} + \int_0^{tH^{-1}(d/n)} H(x) dx. \quad (3.10)$$

By (3.3) and Bingham et al. [13, p. 26] we have for  $0 < \alpha < 1$

$$\lim_{y \rightarrow \infty} \frac{\int_0^y H(x) dx}{yH(y)/(1-\alpha)} = 1, \quad (3.11)$$

and therefore

$$m_n(t) = O\left(H^{-1}(d/n) \frac{d}{n}\right). \quad (3.12)$$

If  $\alpha = 1$ , by assumption  $e_0$  is symmetric, so under (1.1) we have that  $X_1 = e_1 + c_1$  and therefore

$$\begin{aligned} m_n(t) &= O(1) + E[e_0 I\{|X_0| \leq tH^{-1}(d/n)\}] \\ &= O(1) + \int_{tH^{-1}(d/n)-c_1}^{tH^{-1}(d/n)+c_1} x dP\{e_1 \leq x\} \\ &= O\left(H^{-1}(d/n) \frac{d}{n}\right) + \int_{tH^{-1}(d/n)-c_1}^{tH^{-1}(d/n)+c_1} P\{e_1 \leq x\} dx \\ &= O\left(H^{-1}(d/n) \frac{d}{n} \log H^{-1}(d/n)\right). \end{aligned} \quad (3.13)$$

Thus we get from (3.9)–(3.13) for all  $0 < \alpha < 2$  that

$$\frac{nm_n(s)m_n(t)}{A_n^2} \rightarrow 0. \quad (3.14)$$

Lemma 3.3 yields

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{A_n^2} \frac{L(H^{-1}(d/n))}{L_*(H^{-1}(d/n))} E X_0 I\{|X_0| \leq sH^{-1}(d/n)\} X_k I\{|X_k| \leq tH^{-1}(d/n)\} \\ = \frac{\alpha}{2-\alpha} \frac{\rho^k}{1-|\rho|^\alpha} (\min(s, t|\rho|^{-k}))^{2-\alpha}. \end{aligned}$$

By (3.4) we have

$$\lim_{n \rightarrow \infty} \frac{L(H^{-1}(d/n))}{L_*(H^{-1}(d/n))} = \frac{1}{1-|\rho|^\alpha},$$

which completes the proof of the lemma.  $\square$

**Lemma 3.5.** If (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold, we have for all  $1/2 \leq s \leq t \leq 3/2$  and  $0 \leq x \leq 1$  that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{A_n^2} E \left( \sum_{k=1}^{\lfloor nx \rfloor} (u_{k,n}(s) - m_n(s)) \right) \left( \sum_{k=1}^{\lfloor nx \rfloor} (u_{k,n}(t) - m_n(t)) \right) \\ = x \frac{\alpha}{2-\alpha} \left( s^{2-\alpha} + \sum_{k=1}^{\infty} \rho^k [\min(s, t|\rho|^{-k})^{2-\alpha} + \min(t, s|\rho|^{-k})^{2-\alpha}] \right). \end{aligned}$$

**Proof.** We note that

$$\begin{aligned} E\left(\sum_{k=1}^{\lfloor nx \rfloor} (u_{k,n}(s) - m_n(s))\right) & \left(\sum_{k=1}^{\lfloor nx \rfloor} (u_{k,n}(t) - m_n(t))\right) \\ & = \lfloor nx \rfloor E(u_{0,n}(s) - m_n(s))(u_{0,n}(t) - m_n(t)) \\ & \quad + \sum_{k=1}^{\lfloor nx \rfloor - 1} (\lfloor nx \rfloor - k) E(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t)) \\ & \quad + \sum_{k=1}^{\lfloor nx \rfloor - 1} (\lfloor nx \rfloor - k) E(u_{0,n}(t) - m_n(t))(u_{k,n}(s) - m_n(s)). \end{aligned}$$

Let

$$e_k^* = \sum_{\ell=0}^{k-1} \rho^\ell \varepsilon_{k-\ell} \quad \text{and} \quad X_k^* = c_1 + e_k^*. \quad (3.15)$$

Using again Theorem 2.3 of Cline [16] (cf. also [25]), it follows that there is a constant  $C_1$  such that

$$P\{|X_k^*| > x\} \leq C_1 x^{-\alpha} L(x) \quad \text{for all } k \text{ and } 0 \leq x < \infty. \quad (3.16)$$

Clearly as in (3.5),

$$X_k - X_k^* = e_k - e_k^* = \sum_{\ell=k}^{\infty} \rho^\ell \varepsilon_{k-\ell} = \sum_{j=0}^{\infty} \rho^{k+j} \varepsilon_{-j} = \rho^k (X_0 - c_1). \quad (3.17)$$

Next we write

$$\begin{aligned} |E(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))| & = |Eu_{0,n}(t)u_{0,n}(s) - m_n(t)m_n(s)| \\ & \leq |E(X_0(X_k - X_k^*))I\{|X_0| \leq sH^{-1}(d/n)\}I\{|X_k| \leq tH^{-1}(d/n)\}| \\ & \quad + |E(X_0X_k^*)I\{|X_0| \leq sH^{-1}(d/n)\}I\{|X_k| \leq tH^{-1}(d/n)\} - m_n(s)m_n(t)| \\ & \leq A_{1,k,n} + A_{2,k,n} + A_{3,k,n} \end{aligned}$$

with

$$\begin{aligned} A_{1,k,n} & = E|X_0(X_k - X_k^*)I\{|X_0| \leq sH^{-1}(d/n)\}I\{|X_k| \leq tH^{-1}(d/n)\}|, \\ A_{2,k,n} & = E[|X_0X_k^*I\{|X_0| \leq sH^{-1}(d/n)\}I\{|X_k| \leq tH^{-1}(d/n)\} \\ & \quad - I\{|X_k^*| \leq tH^{-1}(d/n)\}|] \end{aligned}$$

and

$$A_{3,k,n} = |E(X_0X_k^*)I\{|X_0| \leq sH^{-1}(d/n)\}I\{|X_k^*| \leq tH^{-1}(d/n)\} - m_n(s)m_n(t)|.$$

Using (3.14) and (3.17) we conclude

$$\begin{aligned} A_{1,k,n} & \leq |\rho|^k E|X_0| |X_0 - c_1| I\{|X_0| \leq sH^{-1}(d/n)\} \\ & \leq C_2 |\rho|^k (H^{-1}(d/n))^2 d/n \end{aligned} \quad (3.18)$$

with some constant  $C_2$ . Next we note that

$$\begin{aligned} A_{2,k,n} & \leq E[|X_0X_k^*I\{|X_0| \leq sH^{-1}(d/n)\}I\{tH^{-1}(d/n)\} \\ & \quad - |\rho|^k |X_0| \leq |X_k^*| \leq tH^{-1}(d/n)] \end{aligned}$$

$$\begin{aligned}
& + E[|X_0 X_k^*| I\{|X_0| \leq sH^{-1}(d/n)\} I\{tH^{-1}(d/n) \\
& \leq |X_k^*| \leq tH^{-1}(d/n) + |\rho|^k |X_0|\}] \\
& = A_{2,k,n}^{(1)} + A_{2,k,n}^{(2)}. \tag{3.19}
\end{aligned}$$

Using the independence of  $X_0$  and  $X_k^*$  we get

$$\begin{aligned}
A_{2,k,n}^{(1)} & \leq E|X_0| I\{|X_0| \leq sH^{-1}(d/n)\} E|X_k^*| I\{tH^{-1}(d/n) - |\rho|^k H^{-1}(d/n) \\
& \leq |X_k^*| \leq tH^{-1}(d/n)\}.
\end{aligned}$$

**Q4** By (3.16) we have that

$$\begin{aligned}
& E|X_k^*| I\{tH^{-1}(d/n) - |\rho|^k H^{-1}(d/n) \leq |X_k^*| \leq tH^{-1}(d/n)\} \\
& = -x P\{|X_k^*| > x\} \Big|_{tH^{-1}(d/n) - |\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)} + \int_{tH^{-1}(d/n) - |\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)} P\{|X_k^*| > x\} dx \\
& \leq \int_{tH^{-1}(d/n) - |\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)} P\{|X_k^*| > x\} dx \\
& \leq C_3 |\rho|^k H^{-1}(d/n) d/n, \tag{3.20}
\end{aligned}$$

where  $C_3$  is a constant. Hence, on account of (3.9), (3.12) and (3.13) we obtain that with some constant  $C_4$

$$A_{2,k,n}^{(1)} \leq C_4 \rho^k (H^{-1}(d/n))^2 d/n$$

and similarly

$$A_{2,k,n}^{(2)} \leq C_4 \rho^k (H^{-1}(d/n))^2 d/n,$$

resulting in

$$A_{2,k,n} \leq C_5 \rho^k (H^{-1}(d/n))^2 d/n. \tag{3.21}$$

Using again the independence of  $X_0$  and  $X_k^*$  we get

$$A_{3,k,n} = |m_n(s)| |EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} - m_n(t)|.$$

**Q5** It is easy to see that

$$\begin{aligned}
& EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} = EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} I\{|X_0| > |\rho|^{-k/2} H^{-1}(d/n)\} \\
& + EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} I\{|X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}
\end{aligned}$$

and by the independence of  $X_0$  and  $X_k^*$  and (3.16) we have

$$\begin{aligned}
& |EX_k^* I\{|X_k^*| \leq tH^{-1}(d/n)\} I\{|X_0| > |\rho|^{-k/2} H^{-1}(d/n)\}| \\
& \leq C_5 |m_n(t)| H(|\rho|^{-k/2} H^{-1}(d/n)) \\
& \leq C_6 |m_n(t)| |\rho|^{k\alpha/2} d/n.
\end{aligned}$$

Next we note that

$$\begin{aligned}
& |E[X_k^* I\{|X_k^*| \leq tH^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}] \\
& - E[(X_k^* + \rho^k(X_0 - c_1)) I\{|X_k^* + \rho^k(X_0 - c_1)\} \leq tH^{-1}(d/n), \\
& |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}]|
\end{aligned}$$

$$\begin{aligned}
&\leq |\rho|^k E[|X_0 - c_1| I\{|X_k^* + \rho^k(X_0 - c_1)| \leq tH^{-1}(d/n), \\
&\quad |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}] \\
&\quad + E[|X_k^*| I\{|X_k^*| \leq tH^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\} \\
&\quad - I\{|X_k^* + \rho^k(X_0 - c_1)| \leq tH^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}] \\
&\leq |\rho|^k (|\rho|^{-k/2} H^{-1}(d/n) + |c_1|) \\
&\quad + E|X_k^*| I\{(t - |\rho|^{k/2})H^{-1}(d/n) - |c_1| |\rho|^k \leq |X_k^*| \leq tH^{-1}(d/n)\} \\
&\quad + E|X_k^*| I\{tH^{-1}(d/n) \leq |X_k^*| \leq (t + |\rho|^{-k/2})H^{-1}(d/n) + |c_1| |\rho|^k\} \\
&\leq C_7(|\rho|^{k/2} H^{-1}(d/n) + |\rho|^k H^{-1}(d/n)d/n)
\end{aligned}$$

by (3.20). Similarly

$$\begin{aligned}
&|EX_k I\{|X_k| \leq tH^{-1}(d/n)\} - EX_k I\{|X_k| \leq tH^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}| \\
&\leq C_8(|\rho|^{k/2} H^{-1}(d/n) + |\rho|^k H^{-1}(d/n)d/n).
\end{aligned}$$

Hence

$$A_{3,k,n} \leq C_9 |\rho|^{\tau k} (H^{-1}(d/n))^2 d/n, \quad \text{where } \tau = \min\{1, \alpha\}/2. \quad (3.22)$$

Putting together (3.18), (3.21) and (3.22) we get that

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{A_n^2} \sum_{k=K}^{\lfloor nx \rfloor - 1} |(\lfloor nx \rfloor - k)E(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))| = 0. \quad (3.23)$$

The lemma now follows from Lemma 3.4 and (3.23).  $\square$

#### 4. A weak convergence result

Define the two-parameter process

$$L_n(t, x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx \rfloor} (X_i I\{|X_i| \leq tH^{-1}(d/n)\} - m_n(t)),$$

for  $0 \leq x \leq 1$ ,  $1/2 \leq t \leq 3/2$ . First we show the tightness of  $L_n(t, x)$ . The proof is based on a generalization of [10]. We introduce

$$X_{i,1} = \max(X_i, 0), \quad X_{i,2} = \min(X_i, 0)$$

and

$$m_{n,1}(t) = EX_{0,1} I\{|X_0| \leq tH^{-1}(d/n)\}, \quad m_{n,2}(t) = EX_{0,2} I\{|X_0| \leq tH^{-1}(d/n)\}.$$

Similarly to  $L_n(t, x)$ , we define

$$L_{n,1}(t, x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx \rfloor} (X_{i,1} I\{|X_i| \leq tH^{-1}(d/n)\} - m_{n,1}(t)),$$

and  $L_{n,2}(t, x)$  is defined in a similar fashion. Clearly, if both  $L_{n,1}$  and  $L_{n,2}$  are tight, then  $L_n(t, x)$  is tight as well. We prove only tightness of  $L_{n,1}$ , the same argument can be used in case of  $L_{n,2}$ . Let

$$g_n = \frac{1}{d^{1/2} \log \log n}.$$

**Lemma 4.1.** If (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold, then

$$m_{n,1}(t) \text{ is a non-decreasing function on } [1/2, 3/2], \quad (4.1)$$

$$\frac{n}{A_n} \sup_{|t_2 - t_1| \leq g_n} |m_{n,1}(t_2) - m_{n,1}(t_1)| \rightarrow 0, \quad n \rightarrow \infty, \quad (4.2)$$

$$E|L_{n,1}(t_2, x) - L_{n,1}(t_1, x)|^6 \leq C_1 |t_2 - t_1|^\tau, \quad \text{if } |t_2 - t_1| \geq g_n, \quad (4.3)$$

and

$$E|L_{n,1}(t, x_2) - L_{n,1}(t, x_1)|^6 \leq C_1 |x_2 - x_1|^\tau, \quad \text{if } |x_2 - x_1| \geq g_n, \quad (4.4)$$

with some  $\tau > 2$  and constant  $C_1$ .

**Proof.** The definition of  $m_{n,1}(t)$  implies immediately (4.1).

By the definition of  $m_{n,1}(t)$  we have for all  $1/2 \leq t_1 \leq t_2 \leq 3/2$  that

$$\begin{aligned} 0 &\leq m_{n,1}(t_2) - m_{n,1}(t_1) = EX_{0,1}(I\{t_1 H^{-1}(d/n) < |X_0| \leq t_2 H^{-1}(d/n)\}) \\ &\leq \int_{t_1 H^{-1}(d/n)}^{t_2 H^{-1}(d/n)} x dH(x) \\ &\leq C_2 (|t_2 H^{-1}(d/n) H(t_2 H^{-1}(d/n)) - t_1 H^{-1}(d/n) H(t_1 H^{-1}(d/n))| \\ &\quad + |t_2 - t_1| H^{-1}(d/n) H(t_1 H^{-1}(d/n))) \\ &\leq C_3 |t_2 - t_1| \frac{d}{n} H^{-1}(d/n) \end{aligned}$$

on account of integration by parts and (3.3), establishing (4.2).

Next we introduce

$$Y_i = \sum_{k=0}^{\lfloor K \log n \rfloor} \rho^k \varepsilon_{i-k} + c_1, \quad Y_{i,1} = \max(Y_i, 0) \quad (4.5)$$

and  $\xi_i = \eta_i - E\eta_i$  with

$$\eta_i = \eta_i(t_1, t_2) = Y_{i,1} I\{t_1 H^{-1}(d/n) < |Y_i| \leq t_2 H^{-1}(d/n)\}.$$

Since  $E|\varepsilon_0|^{\alpha/2} < \infty$ , using Markov's inequality we see that for every  $\beta > 0$  there is a constant  $K = K(\beta)$  such that

$$\left| E(L_{n,1}(t_2, x) - L_{n,1}(t_1, x))^6 - \frac{1}{A_n^6} \sum_{1 \leq i_1, \dots, i_6 \leq \lfloor nx \rfloor} E\xi_{i_1} \dots \xi_{i_6} \right| \leq C_5 n^{-\beta}. \quad (4.6)$$

We note that by definition,  $\{\xi_i\}$  is a stationary,  $\lfloor K \log n \rfloor$ -dependent sequence with zero mean. Let us divide the indices  $i_1, \dots, i_6$  into groups so that the difference between the indices within a group is less than  $\lfloor K \log n \rfloor$  and between groups is larger than  $\lfloor K \log n \rfloor$ . Clearly  $E\xi_{i_1} \dots \xi_{i_6} = 0$ , if there is at least one group containing a single element. So it suffices to consider the cases when all groups contain at least two elements. This allows the cases of one single group with 6 elements ( $D_1$ ), two groups with 3 + 3 ( $D_2$ ) or 4 + 2 ( $D_3$ ) elements and finally 3 groups with 2 elements in each ( $D_4$ ). If there is only one group, then via Hölder's inequality we have

$$|E\xi_{i_1} \dots \xi_{i_6}| \leq E|\xi_0|^6 \leq 2^6 (E|\eta_0|^6 + |E\eta_0|^6).$$

Since the cardinality of  $D_1$  is bounded by constant times  $n(\log n)^5$  we conclude

$$\left| \frac{1}{A_n^6} \sum_{D_1} E \xi_{i_1} \dots \xi_{i_6} \right| \leq C_6 \left( \frac{n(\log n)^5}{A_n^6} [EX_0^6 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\}] \right. \\ \left. + (EX_0 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\})^6 \right] + n^{-\beta} \Big).$$

Integration by parts and (3.3) yield

$$EX_0^6 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\} \leq C_7 |t_2 - t_1| \frac{d}{n} (H^{-1}(d/n))^6,$$

resulting in

$$\left| \frac{1}{A_n^6} \sum_{D_1} E \xi_{i_1} \dots \xi_{i_6} \right| \leq C_8 \left( \frac{(\log n)^5}{d^2} |t_2 - t_1| + n^{-\beta} \right).$$

Using again the  $\lfloor K \log n \rfloor$  dependence of  $\{\xi_i\}$  and the fact that the cardinality of  $D_2$  is constant times  $n^2(\log n)^4$  we conclude via Hölder's inequality

$$\left| \frac{1}{A_n^6} \sum_{D_2} E \xi_{i_1} \dots \xi_{i_6} \right| = \left| \frac{1}{A_n^6} \sum_{D_2} E \xi_{i_1} \xi_{i_2} \xi_{i_3} E \xi_{i_4} \xi_{i_5} \xi_{i_6} \right| \\ \leq C_8 \left( \frac{n^2(\log n)^4}{A_n^6} [EX_0^3 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\}] \right. \\ \left. + (EX_0 I\{t_1 H^{-1}(d/n) \leq |X_0| \leq t_2 H^{-1}(d/n)\})^3 \right]^2 + n^{-\beta} \Big) \\ \leq C_9 \left( \frac{(\log n)^4}{d} (t_2 - t_1)^2 + n^{-\beta} \right).$$

Similar arguments give

$$\left| \frac{1}{A_n^6} \sum_{D_3} E \xi_{i_1} \dots \xi_{i_6} \right| \leq C_{10} \left( \frac{(\log n)^4}{d} (t_2 - t_1)^2 + n^{-\beta} \right).$$

Following the proof of Lemma 3.5 we obtain

$$\left| \frac{1}{A_n^6} \sum_{D_4} E \xi_{i_1} \dots \xi_{i_6} \right| \leq C_{11} \left( \frac{1}{A_n^6} \left( n \sum_{i=0}^{\infty} \xi_0 \xi_i \right)^3 + n^{-\beta} \right) \\ \leq C_{11} \left( |t_2 - t_1|^3 + n^{-\beta} \right).$$

Putting together our estimates and using the choice of  $g_n$  we conclude for all  $|t_2 - t_1| \geq g_n$

$$E(L_{n,1}(t_2, x) - L_{n,1}(t_1, x))^6 \leq C_{12} \left( \frac{(\log n)^5}{d^2} |t_2 - t_1| + \frac{(\log n)^4}{d} |t_2 - t_1|^2 \right. \\ \left. + |t_2 - t_1|^3 + n^{-\beta} \right) \\ \leq C_{13} |t_2 - t_1|^\tau$$

with any  $2 < \tau \leq 3$  on account of assumption (2.10). Hence the proof of (4.3) is complete.

The proof of (4.4) goes along the lines of the arguments used to establish (4.3) and therefore it is omitted.  $\square$

**Lemma 4.2.** *If (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold, then  $L_n(t, x)$  is tight in  $\mathcal{D}([1/2, 3/2] \times [0, 1])$ .*

**Proof.** It follows from a minor modification of Lemma 6 in [10] that both  $L_{n,1}$  and  $L_{n,2}$  are tight. Since  $L_n = L_{n,1} + L_{n,2}$ , the result is proven.  $\square$

Next we consider the convergence of the finite dimensional distributions. It is based in the following lemma:

**Lemma 4.3.** *We assume that (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold. Let  $N = \lfloor (\log n)^\gamma \rfloor$  with some  $\gamma > 0$ . Then*

$$\begin{aligned} E \left( \sum_{i=1}^N X_i I\{|X_i| \leq tH^{-1}(d/n)\} - E[X_i I\{|X_i| \leq tH^{-1}(d/n)\}] \right)^4 \\ \leq C_{13} \left( N(\log N)^3 (H^{-1}(d/n))^4 \frac{d}{n} + N^2 (H^{-1}(d/n))^4 \left( \frac{d}{n} \right)^2 \right) \end{aligned} \quad (4.7)$$

with some constant  $C_{13}$  and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{Nn}{A_n^2} E \left( \sum_{k=1}^N (u_{k,n}(s) - m_n(s)) \right) \left( \sum_{k=1}^N (u_{k,n}(t) - m_n(t)) \right) \\ = \frac{\alpha}{2 - \alpha} \left( s^{2-\alpha} + \sum_{k=1}^{\infty} \rho^k [\min(s, t|\rho|^{-k})^{2-\alpha} + \min(t, s|\rho|^{-k})^{2-\alpha}] \right). \end{aligned} \quad (4.8)$$

**Proof.** We recall the definition of  $\xi_i$  from the proof of Lemma 4.1. For any  $\beta > 0$ , choosing  $K$  in the definition of  $Y_i$  in (4.5) we get that

$$\begin{aligned} E \left( \sum_{i=1}^N X_i I\{|X_i| \leq tH^{-1}(d/n)\} - E[X_i I\{|X_i| \leq tH^{-1}(d/n)\}] \right)^4 \\ \leq C_{14} \left( E \left( \sum_{i=1}^N \xi_i \right)^4 + n^{-\beta} \right). \end{aligned}$$

We write

$$E \left( \sum_{i=1}^N \xi_i \right)^4 = \sum_{i_1, \dots, i_4} E \xi_{i_1} \dots \xi_{i_4}.$$

We note again that the  $\{\xi_i\}$  is a stationary  $K \log n$  dependent sequence with 0 mean. Let us divide the indices  $i_1, \dots, i_4$  into blocks so that the difference between the indices within a block is less than  $K \log n$  and between blocks is larger than  $K \log n$ . Clearly  $E \xi_{i_1} \dots \xi_{i_4} = 0$ , if there is at least one block containing only a single element. So we need to consider the cases of one single block with 4 elements ( $D_1$ ) and two blocks with  $2 + 2$  elements ( $D_2$ ). The number of the elements in  $D_1$  is not greater than constant times  $N(\log N)^3$  and as we showed in the proof of Lemma 4.1

$$E \xi_0^4 \leq C_{14} \left( (H^{-1}(d/n))^4 \frac{d}{n} + n^{-\beta} \right),$$



assuming that  $K$  in (4.5) is sufficiently large. Hence

$$\left| \sum_{D_1}^N E \xi_{i_1} \dots \xi_{i_4} \right| \leq C_{15} \left( N(\log N)^3 (H^{-1}(d/n))^4 \frac{d}{n} + n^{-\beta} \right).$$

As in the proof of Lemma 4.1 we get that

$$\left| \sum_{D_2}^N E \xi_{i_1} \dots \xi_{i_4} \right| \leq C_{16} N^2 \left( \sum_{i=0}^2 |E \xi_0 \xi_i| \right)^2$$

and

$$\sum_{i=0}^{\infty} |E \xi_0 \xi_i| \leq \left( C_{17} (H^{-1}(d/n))^2 \frac{d}{n} + n^{-\beta} \right),$$

completing the proof of (4.7). The proof of (4.8) goes along the lines of the arguments used to establish Lemma 3.5.  $\square$

**Lemma 4.4.** *If (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold, then*

$$L_n(t, x) \longrightarrow \Gamma(t, x) \quad \text{weakly in } \mathcal{D}([1/2, 3/2]) \times [0, 1],$$

where  $\Gamma(t, x)$  is a Gaussian process with  $E \Gamma(t, x) = 0$  and

$$E \Gamma(t, x) \Gamma(s, y) = \min(x, y) \frac{\alpha}{2 - \alpha} \left( (\min(s, t))^{2-\alpha} + \sum_{k=1}^{\infty} \rho^k [\min(s, t |\rho|^{-k})^{2-\alpha} + \min(t, s |\rho|^{-k})^{2-\alpha}] \right).$$

**Proof.** By Lemma 4.2, the process  $L_n(t, x)$  is tight, so we need only to show the convergence of the finite dimensional distributions. By the Cramér–Wold device it is sufficient to prove the asymptotic normality of

$$Q_n = \sum_{j=1}^J \sum_{\ell=0}^L \mu_{j,\ell} (L_n(t_j, x_{\ell+1}) - L_n(t_j, x_{\ell}))$$

for all  $J, L$ , real coefficients  $\mu_{j,\ell}$ ,  $1/2 \leq t_j \leq 3/2$ ,  $1 \leq j \leq J$ , and  $0 = x_0 < x_1 < \dots < x_L < x_{L+1} = 1$ . We recall the definition of  $X_k^*$  from the proof of Lemma 3.5 (cf. (3.17)) and define

$$\bar{L}_n(t, x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx \rfloor} (X_k^* I\{|X_k^*| \leq t H^{-1}(d/n)\} - E X_k^* I\{|X_k^*| \leq t H^{-1}(d/n)\}).$$

Choosing  $K$  large enough in the definition of  $X_k^*$ , we get from the arguments used in the proof of Lemmas 3.5, 4.1 and 4.3 that

$$E(L_n(t, x) - \bar{L}_n(t, x))^2 \rightarrow 0.$$

So we need to establish only the asymptotic normality of

$$\bar{Q}_n = \sum_{\ell=0}^L \sum_{j=1}^J \mu_{j,\ell} (\bar{L}_n(t_j, x_{\ell+1}) - \bar{L}_n(t_j, x_{\ell})).$$

1 Let

$$2 \quad z_{k,\ell} = \sum_{j=1}^J \mu_{j,\ell} (X_k^* I\{|X_k^*| \leq t_j H^{-1}(d/n)\} - E[X_k^* I\{|X_k^*| \leq t_j H^{-1}(d/n)\}]).$$

3 Since for all  $\ell$

$$4 \quad E \left( \frac{1}{A_n} \sum_{k=1}^{\lfloor K \log n \rfloor} z_{k,\ell} \right)^2 \rightarrow 0,$$

5 by stationarity and the  $\lfloor K \log n \rfloor$ -dependence of  $z_{k,\ell}$  for any  $\ell$  we get that the variables

$$6 \quad \frac{1}{A_n} \sum_{k=\lfloor nx_\ell \rfloor + 1}^{\lfloor nx_{\ell+1} \rfloor} z_{k,\ell}, \quad 1 \leq \ell \leq L \text{ are asymptotically independent.}$$

7 By stationarity we have

$$8 \quad \frac{1}{A_n} \sum_{k=\lfloor nx_\ell \rfloor + 1}^{\lfloor nx_{\ell+1} \rfloor} z_{k,\ell} \stackrel{\mathcal{D}}{=} \frac{1}{A_n} \sum_{k=1}^{\lfloor nx_{\ell+1} \rfloor - \lfloor nx_\ell \rfloor} z_{k,\ell}.$$

9 Let us divide the integers of  $[1, \lfloor nx_{\ell+1} \rfloor - \lfloor nx_\ell \rfloor]$  into consecutive blocks  $R_1, V_1, R_2, V_2, \dots, R_s,$   
 10  $V_s$  such that for  $1 \leq i \leq s-1$ ,  $R_i$  contains  $\lfloor (\log n)^\gamma \rfloor$  integers,  $V_i$  contains  $\lfloor K \log n \rfloor$  integers,  
 11 the last two blocks might contain less elements. Let

$$12 \quad \zeta_{i,1} = \sum_{k \in R_i} z_{k,\ell} \quad \text{and} \quad \zeta_{i,2} = \sum_{k \in V_i} z_{k,\ell}.$$

13 Due to the  $\lfloor K \log n \rfloor$  dependence and stationarity, the variables  $\zeta_{i,2}$ ,  $1 \leq i < s$  are independent  
 14 and identically distributed and the proof of [Lemma 3.5](#) shows that

$$15 \quad E \left( \frac{1}{A_n} \sum_{i=1}^s \zeta_{i,2} \right)^2 \rightarrow 0.$$

16 Using [Lemma 4.3](#) we get that

$$17 \quad E \zeta_{i,1}^2 \geq C_{18} (\log n)^\gamma (H^{-1}(d/n))^2 d/n$$

18 and

$$19 \quad E \zeta_{i,1}^2 \leq C_{19} \left( (\log n)^\gamma (\log \log n)^3 (H^{-1}(d/n))^4 \frac{d}{n} + (\log n)^{2\gamma} (H^{-1}(d/n))^4 \left( \frac{d}{n} \right)^2 \right).$$

20 Since  $s$  is proportional to  $n/(\log n)^\gamma$ , a simple calculation yields

$$21 \quad \frac{\sum_{i=1}^s E \zeta_{i,1}^4}{\left( \sum_{i=1}^s E \zeta_{i,1}^2 \right)^2} \rightarrow 0.$$

22 Thus the central limit theorem with Lyapunov's remainder term (cf. [44, p. 154]) implies the  
 23 asymptotic normality of  $\sum_{1 \leq k \leq \lfloor nx_{\ell+1} \rfloor - \lfloor nx_\ell \rfloor} z_{k,\ell}$ . This completes the proof of [Lemma 4.4](#).  $\square$

## 5. Proof of Theorems 2.1 and 2.2

We need the weak law of large numbers for  $\eta_{d,n}$ .

**Lemma 5.1.** *If (1.1), (2.1)–(2.4), (2.7), (2.9)–(2.11) and (3.1) hold, then we have*

$$\frac{\eta_{d,n}}{H^{-1}(d/n)} \xrightarrow{P} 1.$$

**Proof.** Using Gorodetskii [30] and Withers [50] we get that  $X_k$  is a strongly mixing stationary sequence with mixing rate  $\alpha(k) \leq C_1 \exp(-\lambda k)$  for some  $C_1 > 0$  and  $\lambda > 0$ . Fix  $1/2 < t < 2$  and let  $T_k = I\{|X_k| \geq tH^{-1}(d/n)\}$ ,  $1 \leq k \leq n$ . Clearly,  $ET_k = P\{|X_k| \geq tH^{-1}(d/n)\} = H(tH^{-1}(d/n))$  and due to the regular variation of  $H$ ,  $ET_k/(d/n) \rightarrow t^{-\alpha}$ , as  $n \rightarrow \infty$ . On the other hand, by the correlation inequality of Davydov [26] we get for any  $p > 2$  that

$$\begin{aligned} |ET_0T_k - ET_0ET_k| &\leq (\alpha(k))^{(p-1)/p} (ET_0^p)^{1/p} (ET_k^p)^{1/p} \\ &\leq C_1 \exp(-\lambda k(p-1)/p) (ET_0^p)^{2/p} \\ &= C_1 \exp(-\lambda k(p-1)/p) (ET_0)^{2/p} \\ &\leq C_2 \exp(-\lambda k(p-1)/p) (d/n)^{2/p}. \end{aligned}$$

Hence setting  $\bar{T}_k = T_k - ET_k$  we conclude that

$$\begin{aligned} E \left( \sum_{k=1}^n \bar{T}_k \right)^2 &= nE\bar{T}_0^2 + 2 \sum_{k=1}^{n-1} (n-k) E\bar{T}_0\bar{T}_k \\ &\leq n \left( E\bar{T}_k^2 + 2 \sum_{k=1}^{n-1} |E\bar{T}_0\bar{T}_k| \right) \\ &\leq n \left( ET_0^2 + C_3 \sum_{k=1}^{n-1} \exp(-\lambda k(p-1)/p) (d/n)^{2/p} \right) \\ &\leq n \left( ET_0 + C_5 (d/n)^{2/p} \right) \\ &\leq n(d/n)^{2/p}. \end{aligned}$$

Thus by Markov's inequality we have that

$$P \left\{ \sum_{k=1}^n \bar{T}_k \geq d^{2/p} \right\} \leq C_6 n^{(p-2)/p} / d^{2/p} \rightarrow 0,$$

provided that  $d/n^{(2-p)/p} \rightarrow 0$ . Since  $d \geq n^\delta$ , choosing  $p$  near 2, it follows that

$$\sum_{k=1}^n T_k = t^{-\alpha} d(1 + o_P(1)) + o_P(d^{2/p}) = t^{-\alpha} d(1 + o_P(1)).$$

In other words,

$$\frac{1}{d} \#\{k \leq n : |X_k| \geq tH^{-1}(d/n)\} \xrightarrow{P} t^{-\alpha}, \quad \text{as } n \rightarrow \infty.$$

This shows that

$$\lim_{n \rightarrow \infty} P\{\eta_{n,d} \geq tH^{-1}(d/n)\} = 1 \quad \text{for } t < 1$$

and

$$\lim_{n \rightarrow \infty} P\{\eta_{n,d} \geq tH^{-1}(d/n)\} = 0 \quad \text{for } t > 1,$$

completing the proof of Lemma 5.1.  $\square$

**Proof of Theorem 2.1.** We note that  $\Gamma(t, x)$  is a continuous process. Hence combining Lemmas 4.4 and 5.1 we conclude

$$L_n(\eta_{d,n}/H^{-1}(d/n), x) \xrightarrow{\mathcal{D}[0,1]} \Gamma(1, x).$$

It is easy to see that

$$\{\Gamma(1, x), 0 \leq x \leq 1\} \stackrel{\mathcal{D}}{=} \left\{ \left( \frac{\alpha}{2-\alpha} \frac{1+\rho}{1-\rho} \right)^{1/2} W(x), 0 \leq x \leq 1 \right\},$$

where  $W(x)$  is a Wiener process, which completes the proof.  $\square$

**Proof of Theorem 2.2.** Since

$$\frac{1}{A_n} T_{n,d}(x) = L_n(\eta_{d,n}/H^{-1}(d/n), x) - \frac{\lfloor nx \rfloor}{n} L_n(\eta_{d,n}/H^{-1}(d/n), 1).$$

Theorem 2.1 yields

$$\frac{1}{A_n} T_{n,d}(x) \xrightarrow{\mathcal{D}[0,1]} \left( \frac{\alpha}{2-\alpha} \frac{1+\rho}{1-\rho} \right)^{1/2} (W(x) - xW(1)).$$

By definition,  $B(x) = W(x) - xW(1)$ ,  $0 \leq x \leq 1$  is a Brownian bridge, so the proof of Theorem 2.2 is complete.  $\square$

## Uncited references

Q9

[47].

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