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Dynamic fundamental solutions for compressible and incompressible modeled poroelastic continua

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Abstract

In a two-phase material not only each constituent, the solid and the fluid, may be compressible on the microscopic level but also the skeleton itself possesses a structural compressibility. If the compression modulus of a constituent is much larger than the compression modulus of the bulk material this constituent is assumed to be materially incompressible. A common example is soil. Governing equations for such materials are found in the framework of Biot's theory either for the unknowns solid displacement and pore pressure or for the unknowns solid displacement and fluid displacement.

For both formulations fundamental solutions are derived using the method of Hörmander. Unlike the u_i^s–p formulation, where the incompressible model can be obtained applying a limiting procedure directly to the compressible system of equations, a complete new derivation is necessary for the u_i^s–u_i^f formulation. This yields a model of incompressibility different from that of the u_i^s–p formulation which seems to be not suitable for poroelastodynamic problems.

Keywords: Poroelasticity; Biot's theory; Incompressible constituents

1. Introduction

A historical review on the subject of multiphase continuum mechanics identifies two poroelastic theories which have been developed and are used nowadays, namely Biot's theory and the Theory of Porous Media. For more details, the reader is directed to the work of de Boer and Ehlers (1988, 1990) or to the recently published monograph (de Boer, 2000).

Based on the work of von Terzaghi, a theoretical description of porous materials saturated by a viscous fluid was presented by Biot (1941). The dynamic extension was done in two papers, one for the low frequency range (Biot, 1956a) and the other for the high frequency range (Biot, 1956b). Based on the work of Fillunger, the Theory of Porous Media has been developed. This theory is based on the axioms of continuum theories of mixtures (Truesdell and Toupin, 1960; Bowen, 1976) extended by the concept of volume fractions by Bowen (1980, 1982) and others (Ehlers, 1993a,b). Remarks on the equivalence of both theories

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are found in the work of Bowen (1982), Ehlers and Kubik (1994) and Schanz and Diebels (2003). In all these publications, linear version of both theories are compared and, finally, the equivalence can only be shown if Biot's apparent mass density is set to zero. More important for the paper at hand, in Schanz and Diebels (2003) it is shown that the differential operator for both theories is equivalent. Therefore, in the following, it is sufficient to discuss the fundamental solutions only for one of both theories. The result is simply transformed to the other theory by changing some material constants. As Biot's theory is more common this theory is used here.

Wave propagation phenomena are often observed in semi-infinite media, e.g., earthquake motion or propagation of machine foundation excitations in the soil and their effect on neighboring buildings. The efficiency of the Boundary Element Method (BEM) in dealing with such semi-infinite domain problems, e.g., soil–structure interaction, have long been recognized by researchers and engineers. However, a mandatory requirement for every boundary element formulation is the knowledge of fundamental solutions. These solutions solve the underlying differential equation with the inhomogeneity of a Dirac distribution. Physically spoken, the response of a system due to a unit impulse is looked for. These solutions exist for a lot of linear problems.

A poroelastic modeled continuum is described by a set of coupled differential equations where two possible choices of unknowns are used. Either, the solid displacement and fluid displacement are chosen which will, in the following, be denoted u_i^s–u_i^f formulation or the solid displacement and the pore pressure are chosen which will be denoted u_i^s–p formulation. As in any time dependent problem the governing equations may be formulated in frequency or Laplace domain or directly in time domain. The latter is the more complicated case because then a hyperbolic system has to be solved contrary to the elliptic system in the transformed domain.

In case of consolidation processes, a quasi-static theory is sufficient. For this special case, a survey of fundamental solutions is given in Cheng and Detournay (1998). But, for treating wave propagation problems a full dynamic model is required. In this case, first approaches to develop fundamental solutions was made by Burridge and Vargas (1979) for the u_i^s–u_i^f formulation. As inhomogeneity they chose only a point force in the solid which is not sufficient for the usage of such a fundamental solution in a BE formulation. Later, Norris (1985) derived time harmonic fundamental solutions for the same formulation using a point force in the solid as well as a point force in the fluid as load. He also obtained explicit asymptotic approximations for far-field displacements, as well as those for low and high frequency responses. For the same set of unknowns but in Laplace domain Manolis and Beskos (1989) published fundamental solutions (see also the corrections in Manolis and Beskos (1990)). Additionally to the derivation of these solutions they pointed out the analogy between poroelasticity and thermoelasticity. However, this analogy is only possible for the u_i^s–p formulation. This was also shown by Bonnet (1987) when he presented the fundamental solution for the u_i^s–p formulation in frequency domain. Additionally, to the three-dimensional (3-d) solutions which he converted from the thermoelastic solutions from Kupradze (1965) he has given the two-dimensional (2-d) solutions. Further, he concluded that the u_i^s–p formulation is sufficient and the u_i^s–u_i^f formulation is overdetermined. In the following, here, this statement is confirmed. It should be mentioned, however, that in Bonnet's paper (Bonnet, 1987) there is some confusing regarding the sign of the time variation assumed for the harmonic variables which in the poroelastic equations is different to that of the thermoelastic ones. This is corrected by Domínguez (1991, 1992). Boutin et al. (1987) published fundamental solutions for Biot's theory but they neglect the inertia terms of the fluid. The respective governing equations are motivated by a homogenization process (Auriault et al., 1985).

With one exception in all above cited papers fundamental solutions are given in transformed domains. A time domain fundamental solution was presented by Norris (1985) and Wiebe and Antes (1991) for the u_i^s–u_i^f formulation. However, in these solutions the viscous coupling of the solid and fluid is neglected. Without this restriction Chen presented in two papers, for a 2-d continuum (Chen, 1994a) and a 3-d continuum (Chen, 1994b), fundamental solutions for the u_i^s–p formulation. These solutions are achieved from the corrected Laplace domain solutions of Bonnet (1987) by inverse transformation resulting partly in an integral which must be solved numerically.

The above cited fundamental solutions are mainly derived by two methods: First, there is the possibility two split the operator by introducing three potentials or, second, to reduce the highly complicated differential operator matrix to a simple scalar operator by the use of the method of Hörmander (1963). The latter is also used here to derive fundamental solutions for both formulations, i.e., the u_i^s–u_i^f formulation and the u_i^s–p formulation.

Beside the compressibility of the constituents also a structural compressibility exists and is modeled in Biot's theory. For some materials, e.g., soil, the compression modulus of each constituent itself is much larger than the compression modulus of the structure. In these cases, it is sufficient to approximate both the fluid and solid constituents as incompressible, i.e., only the structural compressibility remains. For this case, up to now, no fundamental solutions are available in the literature.

In the following, first, Biot's constitutive equations are recalled and the assumptions for incompressibility are given. With these the governing equations for the u_i^s–u_i^f formulation and the u_i^s–p formulation are derived for compressible as well as the special case of incompressible constituents. Subsequently, the fundamental solutions in Laplace domain are derived for both formulations. The fundamental solutions for the compressible case are recalled not only for completeness also to show how the physical approximation of incompressibility is represented in the mathematics of the formulas. As these solutions are the basis of BE formulations also their singular behavior is discussed. Finally, a visualization of the fundamental solutions is presented.

Throughout this paper, the summation convention is applied over repeated indices and Latin indices receive the values 1, 2, and 1, 2, 3 in two-dimensions (2-d) and three-dimensions (3-d), respectively. Commas $()_{,i}$ denote spatial derivatives and dots $\dot{( )}$ denote the time derivative. As usual, the Kronecker delta is denoted by δ_{ij}.

2. Biot's theory—governing equations

Following Biot's approach to model the behavior of porous media, an elastic skeleton with a statistical distribution of interconnected pores is considered (Biot, 1955). This porosity is denoted by

$$\phi = \frac{V^f}{V}, \tag{1}$$

where V^f is the volume of the interconnected pores contained in a sample of bulk volume V. Contrary to these pores the sealed pores will be considered as part of the solid. Full saturation is assumed leading to $V = V^f + V^s$ with V^s the volume of the solid, i.e., a two-phase material is given.

2.1. Constitutive assumptions

If the constitutive equations are formulated for the elastic solid and the interstitial fluid, a partial stress formulation is obtained (Biot, 1955)

$$\sigma_{ij}^s = 2G\varepsilon_{ij}^s + \left(K - \frac{2}{3}G + \frac{Q^2}{R}\right)\varepsilon_{kk}^s\delta_{ij} + Q\varepsilon_{kk}^f\delta_{ij}, \tag{2a}$$

$$\sigma^f = -\phi p = Q\varepsilon_{kk}^s + R\varepsilon_{kk}^f, \tag{2b}$$

with $()^s$ and $()^f$ indicating either solid or fluid, respectively. The respective stress tensor is denoted by σ_{ij}^s and σ^f and the corresponding strain tensor by ε_{ij}^s and ε_{kk}^f. The elastic skeleton is assumed to be isotropic and homogeneous where the two elastic material constants compression modulus K and shear modulus G refer to the bulk material. The coupling between the solid and the fluid is characterized by the two parameters Q and R. In the above, the sign conventions for stress and strain follow that of elasticity, namely, tensile stress and strain is denoted positive. Therefore, in Eq. (2b) the pore pressure, p, is the negative hydrostatic stress in the fluid σ^{f}.

An alternative representation of the constitutive equations (2) are used in Biot's earlier work (Biot, 1941). There, the total stress $\sigma_{ij} = \sigma^{\mathrm{s}}_{ij} + \sigma^{\mathrm{f}}\delta_{ij}$ is introduced and with Biot's effective stress coefficient $\alpha = \phi(1 + Q/R)$ the constitutive equation with the solid strain, $\varepsilon^{\mathrm{s}}_{ij}$, and the pore pressure, p

$$\sigma_{ij} = 2G\varepsilon^{\mathrm{s}}_{ij} + \left(K - \frac{2}{3}G\right)\varepsilon^{\mathrm{s}}_{kk}\delta_{ij} - \alpha\delta_{ij}p, \tag{3a}$$

is obtained. Additional to the total stress σ_{ij}, as a second constitutive equation, the variation of fluid volume per unit reference volume ζ is introduced

$$\zeta = \alpha\varepsilon^{\mathrm{s}}_{kk} + \frac{\phi^2}{R}p. \tag{3b}$$

This variation of fluid ζ is defined by the mass balance over a reference volume, i.e., by the continuity equation

$$\dot{\zeta} + q_{i,i} = a \tag{4}$$

with the specific flux $q_i = \phi\left(\dot{u}^{\mathrm{f}}_I – \dot{u}^{\mathrm{s}}_i\right)$ and a source term $a(t)$. Eq. (4) identify ζ as a kind of strain describing the motion of the fluid relative to the solid which takes a source in the fluid into account. This source term is not motivated by any physical reason but it is later needed for the derivation of the fundamental solutions.

In a two-phase material not only each constituent, the solid and the fluid, may be compressible on a microscopic level but also the skeleton itself possesses a structural compressibility. If the compression modulus of one constituent is much larger on the microscale than the compression modulus of the bulk material this constituent is assumed to be materially incompressible. A common example for a materially incompressible solid constituent is soil. In this case, the individual grains are much stiffer than the skeleton itself. The respective conditions for such incompressibilities are (Detournay and Cheng, 1993)

$$\frac{K}{K^{\mathrm{s}}} \ll 1 \text{ incompressible solid}, \quad \frac{K}{K^{\mathrm{f}}} \ll 1 \text{ incompressible fluid}, \tag{5}$$

where K^{s} denotes the compression modulus of the solid grains and K^{f} the compression modulus of the fluid. With these conditions it is obvious that three cases exists: (i) Only the solid is incompressible. (ii) Only the fluid is incompressible. (iii) Or the combination of both.

To find the respective constitutive equations for each of these cases the material parameters α, R, and Q have to be rewritten in a different way. Considerations of constitutive relations at micromechanical level as given in Detournay and Cheng (1993) lead to a more rational model for this purpose:

$$\alpha = 1 - \frac{K}{K^{\mathrm{s}}}, \tag{6a}$$

$$R = \frac{\phi^2 K^{\mathrm{f}} {K^{\mathrm{s}}}^2}{K^{\mathrm{f}}(K^{\mathrm{s}} - K) + \phi K^{\mathrm{s}}(K^{\mathrm{s}} - K^{\mathrm{f}})}, \tag{6b}$$

$$Q = \frac{\phi(\alpha - \phi) K^{\mathrm{f}} {K^{\mathrm{s}}}^2}{K^{\mathrm{f}}(K^{\mathrm{s}} - K) + \phi K^{\mathrm{s}}(K^{\mathrm{s}} - K^{\mathrm{f}})}. \tag{6c}$$

Inserting in Eqs. (6) the conditions of incompressibility (5) the three different cases are found:

- *Incompressible solid* $K/K^s \ll 1$

$$\alpha \approx 1, \quad R \approx K^f \phi, \quad Q \approx K^f(1-\phi). \tag{7}$$

These limiting values can be inserted in the constitutive assumptions (2) or (3), respectively.

- *Incompressible fluid* $K/K^f \ll 1$

$$\alpha \text{ unchanged}, \quad R \approx \frac{\phi^2 K^s}{1-\phi-\frac{K}{K^s}}, \quad Q \approx \frac{\phi(\alpha-\phi)K^s}{1-\phi-\frac{K}{K^s}}. \tag{8}$$

Also in this case, these limiting values can be inserted in the constitutive assumptions (2) or (3), respectively.

- *Both constituents are assumed to be incompressible* $K/K^s \ll 1$ *and* $K/K^f \ll 1$

$$\alpha \approx 1, \quad R \to \infty, \quad Q \to \infty, \quad \text{but} \quad \frac{Q}{R} = \frac{1-\phi}{\phi}. \tag{9}$$

The relation $R, Q \to \infty$ expresses that the value of R, Q becomes large, however, due to physical reasons it is in any case limited. But, the condition that R becomes large is used to neglect in (3b) the influence of the pore pressure. This condition and $\alpha = 1$ result in the incompressible constitutive assumptions

$$\sigma_{ij} = 2G\varepsilon^s_{ij} + \left(K - \frac{2}{3}G\right)\varepsilon^s_{kk}\delta_{ij} - \delta_{ij}p, \tag{10a}$$

$$\zeta = \varepsilon^s_{kk} \tag{10b}$$

for the total stress formulation. From (10), it is obvious that this special modeling of a porous continuum relates the variation of fluid volume directly to the volumetric solid strain and the pore pressure is added to the solid stress linearly without the weighting factor α.

For the partial stress formulations (2), a different point of view must be considered because inserting the infinite values of Q and R in the constitutive laws (2) result in an infinite stress. Biot (1955) has given as condition for incompressible constituents

$$(1-\phi)\varepsilon^s_{kk} + \phi\varepsilon^f_{kk} = 0, \tag{11}$$

i.e., it is assumed that the dilatation of the bulk material vanishes. Realizing the relation

$$\frac{Q}{R} = \frac{1-\phi}{\phi} \Rightarrow \frac{Q}{R}\varepsilon^s_{kk} + \varepsilon^f_{kk} = 0 \tag{12}$$

also in the partial stress formulation the case of incompressible constituents can be included resulting in the constitutive assumptions

$$\sigma^s_{ij} = 2G\varepsilon^s_{ij} + \left(K - \frac{2}{3}G\right)\varepsilon^s_{kk}\delta_{ij}, \tag{13a}$$

$$\sigma^f = -\phi p = R\left(\frac{Q}{R}\varepsilon^s_{kk} + \varepsilon^f_{kk}\right) \stackrel{!}{=} 0. \tag{13b}$$

To achieve the zero value in Eq. (13b), the condition that the value R becomes large but is limited must be used.

Contrary to the incompressible model formulated for the total stress formulation (10), in the partial stress formulation the assumption of incompressibility (11) results in an uncoupling of the solid and the fluid in the constitutive assumptions. Therefore, the two incompressible models (10) and (13) are different whereas the underlying compressible models (3) and (2), respectively, are identical. This is not really a contradiction. Keeping in mind that an incompressible model is always an approximation for the more realistic compressible case, it is clear that different approximations can exist. However, the question which approximation is better can only be answered by the respective application.

Aiming at the equation of motion to model wave propagation phenomena, it is sufficient to formulate a linear kinematic equation. Hence, in the following, the relation of the solid/fluid strain to the solid/fluid displacement is chosen linear, respectively:

$$\varepsilon_{ij}^{\mathrm{s}} = \frac{1}{2}\left(u_{i,j}^{\mathrm{s}} + u_{j,i}^{\mathrm{s}}\right), \quad \varepsilon_{kk}^{\mathrm{f}} = u_{k,k}^{\mathrm{f}} \tag{14}$$

assuming small deformation gradients.

2.2. Governing equations: compressible model

In the preceding section, the constitutive equations and the kinematic relations have been given. The next step is to state the balances of momentum. In any two-phase material there are three possibilities to formulate the balances of momentum: first, the balance of momentum in the solid; second, the balance of momentum in the fluid; and third, the balance of momentum for the bulk material. But, it is sufficient to choose two of them.

The first to balances are used by Biot (1956a) using the solid displacement and the fluid displacement as unknowns

$$\sigma_{ij,j}^{\mathrm{s}} + (1-\phi)f_i^{\mathrm{s}} = (1-\phi)\rho_{\mathrm{s}}\ddot{u}_i^{\mathrm{s}} + \rho_{\mathrm{a}}\left(\ddot{u}_i^{\mathrm{s}} - \ddot{u}_i^{\mathrm{f}}\right) + \frac{\phi^2}{\kappa}\left(\dot{u}_i^{\mathrm{s}} - \dot{u}_i^{\mathrm{f}}\right), \tag{15a}$$

$$\sigma_i^{\mathrm{f}} + \phi f_i^{\mathrm{f}} = \phi\rho_{\mathrm{f}}\ddot{u}_i^{\mathrm{f}} - \rho_{\mathrm{a}}\left(\ddot{u}_i^{\mathrm{s}} - \ddot{u}_i^{\mathrm{f}}\right) - \frac{\phi^2}{\kappa}\left(\dot{u}_i^{\mathrm{s}} - \dot{u}_i^{\mathrm{f}}\right). \tag{15b}$$

The first balance equation (15a) is that for the solid skeleton and the second (15b) is that for the interstitial fluid. In Eq. (15), the body forces in the solid skeleton f_i^{s} and in the fluid f_i^{f} are introduced. Further, the respective densities are denoted by ρ_{s} and ρ_{f}. To describe the dynamic interaction between fluid and skeleton an additional density the apparent mass density, ρ_{a}, has been introduced by Biot (1956a). It can be written as $\rho_{\mathrm{a}} = C\phi\rho_{\mathrm{f}}$, where C is a factor depending on the geometry of the pores and the frequency of excitation. At low frequency, Bonnet and Auriault (1985) measured $C = 0.66$ for a sphere assembly of glass beads. In higher frequency ranges, a certain functional dependence of C on frequency has been proposed based on conceptual porosity structures, e.g., in Biot (1956b) and Bonnet and Auriault (1985). The factor ϕ^2/κ in front of the damping term is usually denoted by b. Here, the simplification of a frequency independent, respectively time independent, value is taken which is only valid in low frequency range. Further, the above chosen factor ϕ^2/κ is given only in case of circular pores when κ denotes the permeability. However, in the following, any other also frequency dependent factor b could easily be implemented.

The third above mentioned balance of momentum for the mixture is formulated in Biot's earlier work (Biot, 1941) for quasi-statics and in Biot (1956a) for dynamics. This dynamic equilibrium is given by

$$\sigma_{ij,j} + F_i = \rho_{\mathrm{s}}(1-\phi)\ddot{u}_i^{\mathrm{s}} + \phi\rho_{\mathrm{f}}\ddot{u}_i^{\mathrm{f}}, \tag{16}$$

with the bulk body force per unit volume $F_i = (1-\phi)f_i^{\mathrm{s}} + \phi f_i^{\mathrm{f}}$. It is obvious that adding the two partial balances (15a) and (15b) results in the balance of the mixture (16).

In most papers using the total stress formulation, now, the constitutive assumption for the fluid transport in the interstitial space is given by Darcy's law. Here, it is also used, however, with the balance of momentum in the fluid (15b) Darcy's law is already given. Rearranging (15b) and taking the definition of the flux $q_i = \phi\left(\dot{u}_i^{\mathrm{f}} - \dot{u}_i^{\mathrm{s}}\right)$ as well as $\sigma^{\mathrm{f}} = -\phi p$ into account the dynamic version of Darcy's law

$$q_i = -\kappa\left(p_{,i} + \frac{\rho_{\mathrm{a}}}{\phi}\left(\ddot{u}_i^{\mathrm{f}} - \ddot{u}_i^{\mathrm{s}}\right) + \rho_{\mathrm{f}}\ddot{u}_i^{\mathrm{f}} - f_i^{\mathrm{f}}\right) \tag{17}$$

is achieved.

Aiming at the equation of motion, the constitutive equations have to be combined with the corresponding balances of momentum and the kinematic conditions. To do this, first, the degrees of freedom must be determined. There are several possibilities (i) to use the solid displacement u_i^{s} and the fluid displacement u_i^{f} (six unknowns in 3-d) or (ii) a combination of the pore pressure p and the solid displacement u_i^{s} (four unknowns in 3-d). As shown in Bonnet (1987), it is sufficient to use the latter choice. Here, for completeness, both choices will be presented where the first will be denoted by u_i^{s}–u_i^{f} formulation and the latter by u_i^{s}–p formulation.

First, the equations of motion for a poroelastic body are presented for the unknowns solid displacement u_i^{s} and fluid displacement u_i^{f}. Inserting in (15) the constitutive equations (2) written for the partial stress tensors and the linear strain displacement relations (14) yield a set of equations of motion in time domain

$$\begin{aligned} &Gu_{i,jj}^{\mathrm{s}} + \left(K + \frac{1}{3}G\right)u_{j,ij}^{\mathrm{s}} + Q\left(\frac{Q}{R}u_{j,ji}^{\mathrm{s}} + u_{j,ji}^{\mathrm{f}}\right) + (1-\phi)f_i^{\mathrm{s}} \\ &\quad = (1-\phi)\rho_{\mathrm{s}}\ddot{u}_i^{\mathrm{s}} + \rho_{\mathrm{a}}\left(\ddot{u}_i^{\mathrm{s}} - \ddot{u}_i^{\mathrm{f}}\right) + \frac{\phi^2}{\kappa}\left(\dot{u}_i^{\mathrm{s}} - \dot{u}_i^{\mathrm{f}}\right), \end{aligned} \tag{18a}$$

$$R\left(\frac{Q}{R}u_{j,ji}^{\mathrm{s}} + u_{j,ji}^{\mathrm{f}}\right) + \phi f_i^{\mathrm{f}} = \phi\rho_{\mathrm{f}}\ddot{u}_i^{\mathrm{f}} - \rho_{\mathrm{a}}\left(\ddot{u}_i^{\mathrm{s}} - \ddot{u}_i^{\mathrm{f}}\right) - \frac{\phi^2}{\kappa}\left(\dot{u}_i^{\mathrm{s}} - \dot{u}_i^{\mathrm{f}}\right). \tag{18b}$$

Second, the respective equations of motion are presented for the pore pressure p and the solid displacement u_i^{s} as unknowns. To achieve this formulation the fluid displacement u_i^{f} has to be eliminated from Eqs. (3), (4), (16) and (17). In order to do this, Darcy's law (17) is rearranged to obtain u_i^{f}–u_i^{s}. Since this relative displacement is given as second order time derivative in (17) and the flux is related to its first order time derivative by $q_i = \phi\left(\dot{u}_i^{\mathrm{f}}–\dot{u}_i^{\mathrm{s}}\right)$, this is only possible in Laplace domain. After transformation to Laplace domain, the relative fluid to solid displacement is

$$\hat{u}_i^{\mathrm{f}} - \hat{u}_i^{\mathrm{s}} = -\underbrace{\frac{\kappa\rho_{\mathrm{f}}\phi^2 s^2}{\phi^2 s + s^2\kappa(\rho_{\mathrm{a}} + \phi\rho_{\mathrm{f}})}}_{\beta}\frac{1}{s^2\phi\rho_{\mathrm{f}}}\left(\hat{p}_{,i} + s^2\rho_{\mathrm{f}}\hat{u}_i^{\mathrm{s}} - \hat{f}_i^{\mathrm{f}}\right). \tag{19}$$

In Eq. (19), the abbreviation β is defined for further usage and $\mathscr{L}\{f(t)\} = \hat{f}(s)$ denotes the Laplace transform, with the complex variable s. Moreover, vanishing initial conditions for u_i^{s} and u_i^{f} are assumed here and in the following. Now, the final set of differential equations for the displacement $\hat{u}_i^{\mathrm{s}}$ and the pore pressure $\hat{p}$ is obtained by inserting the constitutive equations (3) into the Laplace transformed dynamic equilibrium (16) and continuity equation (4) with $\hat{u}_i^{\mathrm{f}}$–$\hat{u}_i^{\mathrm{s}}$ from Eq. (19). This leads to the final set of differential equations for the displacement $\hat{u}_i^{\mathrm{s}}$ and the pore pressure $\hat{p}$

$$G\hat{u}_{i,jj}^{\mathrm{s}} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ij}^{\mathrm{s}} - (\alpha - \beta)\hat{p}_{,i} - s^2(\rho - \beta\rho_{\mathrm{f}})\hat{u}_i^{\mathrm{s}} = \beta\hat{f}_i^{\mathrm{f}} - \widehat{F}_i, \tag{20a}$$

$$\frac{\beta}{s\rho_{\mathrm{f}}}\hat{p}_{,ii} - \frac{\phi^2 s}{R}\hat{p} - (\alpha - \beta)s\hat{u}_{i,i}^{\mathrm{s}} = -\hat{a} + \frac{\beta}{s\rho_{\mathrm{f}}}\hat{f}_{i,i}^{\mathrm{f}}. \tag{20b}$$

In the above Eqs. (20), the bulk density $\rho = \rho_s(1-\phi) + \phi\rho_f$ is used. This set of equations describe the behavior of a poroelastic continuum completely as well as the u_i^s–u_i^f formulations (18). Contrary to the formulations using the solid and fluid displacement (18) an analytical representation in time domain is only possible for $\kappa \to \infty$. This case would represent a negligible friction between solid and interstitial fluid.

2.3. Governing equations: incompressible model

As mentioned above, often the approximation of incompressible constituents can be used. Regarding the assumption of only one incompressible constituent (7) and (8) no special governing equations must be given because only the material data are changed and not the structure of the constitutive law. So, in the following, the expression 'incompressible' will denote the case when both constituents are modeled incompressible.

In this case of modeling both constituents as incompressible, a different set of governing equations is obtained. Inserting in (18) the incompressibility condition (12), the governing equations are given by

$$Gu^s_{i,jj} + \left(K + \frac{1}{3}G\right)u^s_{j,ij} + (1-\phi)f^s_i = (1-\phi)\rho_s\ddot{u}^s_i + \rho_a\left(\ddot{u}^s_i - \ddot{u}^f_i\right) + \frac{\phi^2}{\kappa}\left(\dot{u}^s_i - \dot{u}^f_i\right), \tag{21a}$$

$$\phi f^f_i = \phi\rho_f\ddot{u}^f_i - \rho_a\left(\ddot{u}^s_i - \ddot{u}^f_i\right) - \frac{\phi^2}{\kappa}\left(\dot{u}^s_i - \dot{u}^f_i\right) \tag{21b}$$

using the solid displacement and fluid displacement as unknowns. In this incompressible version of the equations of motion, the uncoupling of the fluid and solid in the constitutive assumptions is clearly observed as commented in the last section. So, in Eqs. (21) only the coupling by the acceleration and damping terms remains. Further, the second Eq. (21b) is no longer independent. It cannot be used to eliminate the fluid displacement u_i^f in (21a). As an additional equation the incompressibility condition (11) has to be used.

Contrary, if the solid displacement and the pore pressure are used as unknowns a sufficient set of differential equations is obtained. Inserting in (20) simply the conditions (9), i.e., setting $\alpha = 1$ and taking the limit $R \to \infty$, the equations of motion under the assumption of incompressible constituents are achieved resulting in

$$G\hat{u}^s_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}^s_{j,ij} - (1-\beta)\hat{p}_{,i} - s^2(\rho - \beta\rho_f)\hat{u}^s_i = \beta\hat{f}^f_i - \widehat{F}_i, \tag{22a}$$

$$\frac{\beta}{s\rho_f}\hat{p}_{,ii} - (1-\beta)s\hat{u}^s_{i,i} = -\hat{a} + \frac{\beta}{s\rho_f}\hat{f}^f_{i,i}. \tag{22b}$$

The equation for the pore pressure (22b) shows that this variable is no longer a degree of freedom. Integrating of (22b) would yield the gradient of the pore pressure which could then be eliminated in (22a). Physically interpreted the pore pressure is in this case only determined by the deformation of the solid skeleton and no longer by any deformation of the fluid.

3. Fundamental solutions

Fundamental solutions for the above given systems of differential equations are known in closed form only in Fourier domain or Laplace domain. But, even in the transformed domain only the general case of compressible constituents is published. The fundamental solutions for the Laplace transformed system of (18) is given in Manolis and Beskos (1989) and for the Laplace transformed system of (20) in Chen (1994a,b).

Here, the fundamental solutions for the incompressible case are presented. The fundamental solutions for the compressible case are recalled to show how the physical approximation of incompressibility is represented in the mathematical formulas. In order to deduce these solutions, an operator notation is useful. So, for the u_i^s–p formulation the governing equations of the compressible case (20) as well as the incompressible case (22) are reformulated as

$$\mathbf{B4}\begin{bmatrix}\hat{u}_i^s\\ \hat{p}\end{bmatrix}+\begin{bmatrix}\widehat{F}_i\\ \hat{a}\end{bmatrix}=\mathbf{0} \tag{23}$$

with the differential operators

$$\mathbf{B4}^{\text{comp}}=\begin{bmatrix}(G\nabla^2-s^2(\rho-\beta\rho_f))\delta_{ij}+\left(K+\frac{1}{3}G\right)\partial_i\partial_j & -(\alpha-\beta)\partial_i\\ -s(\alpha-\beta)\partial_j & \dfrac{\beta}{s\rho_f}\nabla^2-\dfrac{\phi^2 s}{R}\end{bmatrix}, \tag{24a}$$

$$\mathbf{B4}^{\text{incomp}}=\begin{bmatrix}(G\nabla^2-s^2(\rho-\beta\rho_f))\delta_{ij}+\left(K+\frac{1}{3}G\right)\partial_i\partial_j & -(1-\beta)\partial_i\\ -s(1-\beta)\partial_j & \dfrac{\beta}{s\rho_f}\nabla^2\end{bmatrix}. \tag{24b}$$

In Eqs. (23) and (24), the operator is denoted by **B4** independently whether it is in 2-d ($i,j=1,2$, i.e., three unknowns) or 3-d ($i,j=1,2,3$, i.e., four unknowns). The corresponding representation of a poroelastic continuum using the u_i^s–u_i^f formulation is

$$\mathbf{B6}\begin{bmatrix}\hat{u}_i^s\\ \hat{u}_i^f\end{bmatrix}+\begin{bmatrix}(1-\phi)\hat{f}_i^s\\ \phi\hat{f}_i^f\end{bmatrix}=\mathbf{0} \tag{25}$$

with the differential operators

$$\mathbf{B6}^{\text{comp}}=\begin{bmatrix}B_{ij}^{\text{comp}} & Q\partial_i\partial_j+\left(s^2\rho_a+s\dfrac{\phi^2}{\kappa}\right)\delta_{ij}\\ Q\partial_i\partial_j+\left(s^2\rho_a+s\dfrac{\phi^2}{\kappa}\right)\delta_{ij} & R\partial_i\partial_j-\left(s^2(\phi\rho_f+\rho_a)-s\dfrac{\phi^2}{\kappa}\right)\delta_{ij}\end{bmatrix} \tag{26a}$$

with

$$B_{ij}^{\text{comp}}=\left(G\nabla^2-s^2((1-\phi)\rho_s+\rho_a)-s\frac{\phi^2}{\kappa}\right)\delta_{ij}+\left(K+\frac{1}{3}G+\frac{Q^2}{R}\right)\partial_i\partial_j,$$

$$\mathbf{B6}^{\text{incomp}}=\begin{bmatrix}B_{ij}^{\text{incomp}} & \left(s^2\rho_a+s\dfrac{\phi^2}{\kappa}\right)\delta_{ij}\\ \left(s^2\rho_a+s\dfrac{\phi^2}{\kappa}\right)\delta_{ij} & \left(-s^2(\phi\rho_f+\rho_a)-s\dfrac{\phi^2}{\kappa}\right)\delta_{ij}\end{bmatrix} \tag{26b}$$

with

$$B_{ij}^{\text{incomp}}=\left(G\nabla^2-s^2((1-\phi)\rho_s+\rho_a)-s\frac{\phi^2}{\kappa}\right)\delta_{ij}+\left(K+\frac{1}{3}G\right)\partial_i\partial_j.$$

As before in (24), the operator name **B6** is the same whether it is in 2-d (four unknowns) or 3-d (six unknowns). In the following, the same material parameters in both representations (24) and (26) will be used, so Q is replaced by $Q=R(\alpha/\phi-1)$ to have comparable representations.

In Eqs. (24) and (26), the partial derivative $()_{,i}$ is denoted by ∂_i and $\nabla^2 = \partial_{ii}$ is the Laplacian operator. Note, all the operators (24) and (26) are elliptic but the operators **B6** in (26) are self adjoint whereas the operators **B4** in (24) are not self adjoint. Therefore, in the latter case for the deduction of fundamental solutions the adjoint operator to **B4** has to be used which in the following will not be indicated separately.

A fundamental solution is mathematically spoken a solution of the equation $\mathbf{BG} + \mathbf{I}\delta(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, where the matrix of fundamental solutions is denoted by $\mathbf{G}$, the identity matrix by $\mathbf{I}$, and the Dirac distribution by $\delta(\mathbf{x} - \mathbf{y})$. Physically interpreted the solution at point $\mathbf{x}$ due to a single point force at point $\mathbf{y}$ is looked for. Concerning the interpretation of the 'single point force' the difference in the fundamental solutions for both representations of poroelastic governing equations (23) and (25) becomes obvious. In the system (25), the right hand side consists of forces acting in the solid part $(1-\phi)\hat{f}_i^{\mathrm{s}}$ and in the fluid part $\phi\hat{f}_i^{\mathrm{f}}$ of the porous media, respectively. Contrary, in the system (23), the right hand side consists of a bulk body force $\widehat{F}_i = (1-\phi)\hat{f}_i^{\mathrm{s}}$ and a source term $\hat{a}$, i.e., no forces in the fluid $\hat{f}_i^{\mathrm{f}}$ are present. Due to this, it cannot be expected that the fundamental solutions of both systems coincide. Only the displacement solution due to a single force in the solid will be the same.

To find these solutions, the same method can be chosen for both representations. In all cases, for compressible as well as incompressible constituents and for both representations, respectively, the method of Hörmander (1963) is used. The idea of this method is to reduce the highly complicated operators (24) and (26) to simple well-known operators. For this purpose the definition of the inverse matrix operator $\mathbf{B}^{-1} = \mathbf{B}^{\mathrm{co}}/\det(\mathbf{B})$ with the matrix of cofactors $\mathbf{B}^{\mathrm{co}}$ is used. The ansatz $\mathbf{G} = \mathbf{B}^{\mathrm{co}}\varphi$ for the matrix of fundamental solutions with an unknown scalar function φ inserted in the operator equation $\mathbf{BG} + \mathbf{I}\delta(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ yields a more convenient representation of Eqs. (23) and (25)

$$\mathbf{BB}^{\mathrm{co}}\varphi + \mathbf{I}\delta(\mathbf{x}-\mathbf{y}) = \det(\mathbf{B})\mathbf{I}\varphi + \mathbf{I}\delta(\mathbf{x}-\mathbf{y}) = \mathbf{0} \rightsquigarrow \det(\mathbf{B})\varphi + \delta(\mathbf{x}-\mathbf{y}) = 0. \tag{27}$$

With this reformulation, the search for a fundamental solution is reduced to solve the simpler scalar equation (27). An overview of this method is found in the original work (Hörmander, 1963) and more exemplary in Schanz (2001) and Rashed (2002).

First, this method is applied to the compressible operators in (24).

Compressible model. Following Hörmander's idea, first, the determinants of the operators $\mathbf{B4}^{\mathrm{comp}}$ and $\mathbf{B6}^{\mathrm{comp}}$ are calculated, preferably, with the aid of computer algebra. This yields the results:

2-d :

$$\det(\mathbf{B4}^{\mathrm{comp}}) = \frac{G\beta}{s\rho_{\mathrm{f}}}\left(K + \frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)\left(\nabla^2 - s^2\lambda_1^2\right)\left(\nabla^2 - s^2\lambda_2^2\right), \tag{28}$$

$$\det(\mathbf{B6}^{\mathrm{comp}}) = \frac{-s^2G\phi^2\rho_{\mathrm{f}}}{\beta}\left(K + \frac{4}{3}G\right)R\left(\nabla^2 - s^2\lambda_3^2\right)\left(\nabla^2 - s^2\lambda_1^2\right)\left(\nabla^2 - s^2\lambda_2^2\right), \tag{29}$$

3-d :

$$\det(\mathbf{B4}^{\mathrm{comp}}) = \frac{G^2\beta}{s\rho_{\mathrm{f}}}\left(K + \frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)^2\left(\nabla^2 - s^2\lambda_1^2\right)\left(\nabla^2 - s^2\lambda_2^2\right), \tag{30}$$

$$\det(\mathbf{B6}^{\mathrm{comp}}) = \left(\frac{s^2G\phi^2\rho_{\mathrm{f}}}{\beta}\right)^2\left(K + \frac{4}{3}G\right)R\left(\nabla^2 - s^2\lambda_3^2\right)^2\left(\nabla^2 - s^2\lambda_1^2\right)\left(\nabla^2 - s^2\lambda_2^2\right), \tag{31}$$

with the roots λ_i, $i = 1, 2, 3$

$$\lambda_{1,2}^2 = \frac{1}{2}\left[\frac{\phi^2\rho_f}{\beta R} + \frac{\rho - \beta\rho_f}{K + \frac{4}{3}G} + \frac{\rho_f(\alpha - \beta)^2}{\beta\left(K + \frac{4}{3}G\right)} \pm \sqrt{\left(\frac{\phi^2\rho_f}{\beta R} + \frac{\rho - \beta\rho_f}{K + \frac{4}{3}G} + \frac{\rho_f(\alpha - \beta)^2}{\beta\left(K + \frac{4}{3}G\right)}\right)^2 - 4\frac{\phi^2\rho_f(\rho - \beta\rho_f)}{\beta R\left(K + \frac{4}{3}G\right)}}\right], \tag{32a}$$

$$\lambda_3^2 = \frac{\rho - \beta\rho_f}{G}. \tag{32b}$$

Expressing the determinant using these roots the scalar equation corresponding to (27) is given by

$$\left(\nabla^2 - s^2\lambda_3^2\right)\left(\nabla^2 - s^2\lambda_1^2\right)\left(\nabla^2 - s^2\lambda_2^2\right)\psi + \delta(\mathbf{x} - \mathbf{y}) = 0 \tag{33}$$

using an appropriate abbreviation ψ for every operator, i.e.,

$$\begin{aligned}
&\text{2-d}: \\
&\mathbf{B4}^{\text{comp}}: \psi = G\frac{\beta}{s\rho_f}\left(K + \frac{4}{3}G\right)\varphi, \\
&\mathbf{B6}^{\text{comp}}: \psi = -G\frac{s^2\phi^2\rho_f}{\beta}\left(K + \frac{4}{3}G\right)R\varphi. \\
&\text{3-d}: \\
&\mathbf{B4}^{\text{comp}}: \psi = G^2\frac{\beta}{s\rho_f}\left(K + \frac{4}{3}G\right)(\nabla^2 - s^2\lambda_3^2)\varphi, \\
&\mathbf{B6}^{\text{comp}}: \psi = G^2\left(\frac{s^2\phi^2\rho_f}{\beta}\right)^2\left(K + \frac{4}{3}G\right)R\left(\nabla^2 - s^2\lambda_3^2\right)\varphi.
\end{aligned} \tag{34}$$

The solution of the modified higher order Helmholtz equation (33) is

$$\text{2-d}: \quad \psi = \frac{1}{2\pi s^4}\left[\frac{K_0(\lambda_1 sr)}{\left(\lambda_1^2 - \lambda_2^2\right)\left(\lambda_1^2 - \lambda_3^2\right)} + \frac{K_0(\lambda_2 sr)}{\left(\lambda_2^2 - \lambda_3^2\right)\left(\lambda_2^2 - \lambda_1^2\right)} + \frac{K_0(\lambda_3 sr)}{\left(\lambda_3^2 - \lambda_1^2\right)\left(\lambda_3^2 - \lambda_2^2\right)}\right], \tag{35}$$

$$\text{3-d}: \quad \psi = \frac{1}{4\pi r s^4}\left[\frac{e^{-\lambda_1 sr}}{\left(\lambda_1^2 - \lambda_2^2\right)\left(\lambda_1^2 - \lambda_3^2\right)} + \frac{e^{-\lambda_2 sr}}{\left(\lambda_2^2 - \lambda_1^2\right)\left(\lambda_2^2 - \lambda_3^2\right)} + \frac{e^{-\lambda_3 sr}}{\left(\lambda_3^2 - \lambda_2^2\right)\left(\lambda_3^2 - \lambda_1^2\right)}\right], \tag{36}$$

with the zero order modified Bessel function of second kind $K_0(z)$. Further, the distance between the two points $\mathbf{x}$ and $\mathbf{y}$ is denoted by $r = |\mathbf{x} - \mathbf{y}|$.

Having in mind that the Laplace transformation of the function describing a traveling wave front with constant speed c is $e^{-rs/c} = \mathscr{L}\{H(t - r/c)\}$ (in 3-d), it is obvious that the above solution (36) represents three waves. However, as the roots λ_i are functions of s, here, the wave speeds are time dependent representing the attenuation in a poroelastic continuum. This is in accordance with the well known three wave types of a poroelastic continuum (Biot, 1956a). The roots λ_1, λ_2, and λ_3 correspond to the wave velocities of the slow and fast compressional wave and to the shear wave, respectively. The same is true in 2-d where the damped wave fronts are represented in Laplace domain by the modified Bessel functions $K_0(z)$. It should be remarked that the root λ_3 representing the shear wave is in 3-d a double root whereas it is in 2-d only a single root, which, as in elasticity, corresponds to the number of polarization planes (Royer and Dieulesaint, 2000).

From a pure mathematical point of view, the determinant of the operator $\mathbf{B6}^{\text{comp}}$ can have four roots in 2-d and six roots in 3-d. However, in (29) or (31) only three or four roots are found, respectively. As above discussed, each root represents a different wave type whereas the shear wave corresponds to a double root in 3-d. Therefore, from a physical point of view $\mathbf{B6}^{\text{comp}}$ can be expected to have the same roots as $\mathbf{B4}^{\text{comp}}$, despite the larger matrix dimension. This is confirmed in (29) and (31). As a consequence, it can be concluded that the representation of a poroelastic continuum with solid displacement and fluid displacement is overdetermined, i.e., the representation with pore pressure and solid displacement is sufficient. This confirms the considerations of Bonnet (1987).

The next steps are to insert the solution ψ back in the definition $\mathbf{G} = \mathbf{B}^{\text{co}}\varphi$ taking into account the proper relations (34) between φ and ψ. After calculating the respective matrix of cofactors $\mathbf{B}^{\text{co}}$ the fundamental solutions are found for the u_i^{s}–p formulation

$$\mathbf{G4}^{\text{comp}} = \begin{bmatrix} \widehat{U}_{ij}^{\text{s}} & \widehat{U}_i^{\text{f}} \\ \widehat{P}_j^{\text{s}} & \widehat{P}^{\text{f}} \end{bmatrix} = \frac{s\rho_{\text{f}}}{G\beta(K+\frac{4}{3}G)} \begin{bmatrix} (F\nabla^2 + AD)\delta_{ij} - F\partial_{ij} & -A(\alpha-\beta)s\partial_i \\ -A(\alpha-\beta)\partial_i & A\left(\left(K+\frac{1}{3}G\right)\nabla^2 + A\right) \end{bmatrix}\psi, \tag{37}$$

with the abbreviations

$$A = G\nabla^2 - s^2(\rho - \beta\rho_{\text{f}}), \quad D = \beta/(s\rho_{\text{f}})\nabla^2 - \phi^2 s/R, \quad F = (K + 1/3G)D - (\alpha-\beta)^2 s,$$

and for the u_i^{s}–u_i^{f} formulation

$$\mathbf{G6}^{\text{comp}} = \begin{bmatrix} \widehat{U}_{ij}^{\text{ss}} & \widehat{U}_{ij}^{\text{sf}} \\ \widehat{U}_{ij}^{\text{fs}} & \widehat{U}_{ij}^{\text{ff}} \end{bmatrix} = \frac{-\beta}{Gs^2\phi^2\rho_{\text{f}}\left(K+\frac{4}{3}G\right)} \begin{bmatrix} M_3\partial_{ij} + (M_5 - M_3\nabla^2)\delta_{ij} & M_1\partial_{ij} + (M_4 - M_1\nabla^2)\delta_{ij} \\ M_1\partial_{ij} + (M_4 - M_1\nabla^2)\delta_{ij} & M_2\partial_{ij} + (M_6 - M_2\nabla^2)\delta_{ij} \end{bmatrix}\psi, \tag{38}$$

with the abbreviations

$$M_1 = CE\left[\frac{K+\frac{1}{3}G}{R} + \left(\frac{\alpha}{\phi}-1\right)^2\right] - C^2\left(\frac{\alpha}{\phi}-1\right) + C\nabla^2\left(K+\frac{1}{3}G\right) + B\left[C - E\left(\frac{\alpha}{\phi}-1\right)\right],$$

$$M_2 = 2BC\left(\frac{\alpha}{\phi}-1\right) - B^2 - B\nabla^2\left(K+\frac{1}{3}G\right) - C^2\left[\frac{K+\frac{1}{3}G}{R} + \left(\frac{\alpha}{\phi}-1\right)^2\right],$$

$$M_3 = -E\nabla^2\left(K+\frac{1}{3}G\right) + 2EC\left(\frac{\alpha}{\phi}-1\right) - C^2 - E^2\left[\frac{K+\frac{1}{3}G}{R} + \left(\frac{\alpha}{\phi}-1\right)^2\right],$$

$$M_4 = \frac{s^2\rho_{\text{f}}\phi^2 G}{\beta}(\nabla^2 - s^2\lambda_3^2)\left[\nabla^2\left(\frac{\alpha}{\phi}-1\right) + \frac{C}{R}\right],$$

$$M_5 = \frac{s^2\rho_{\text{f}}\phi^2 G}{\beta}(s^2\lambda_3^2 - \nabla^2)\left[\nabla^2 + \frac{E}{R}\right],$$

$$M_6 = \frac{s^2\rho_{\text{f}}\phi^2 G}{\beta}(s^2\lambda_3^2 - \nabla^2)\left[\nabla^2\left(\frac{K+\frac{1}{3}G}{R} + \left(\frac{\alpha}{\phi}-1\right)^2\right) + \frac{B}{R}\right],$$

$$B = G\nabla^2 - s^2(1-\phi)\rho_s - C, \quad C = s\frac{\phi^2}{\kappa} + s^2\rho_a, \quad E = -s^2\phi\rho_f - C.$$

The difference of the 2-d solution and the 3-d solution lies only in the different functions ψ from (35) or (36), respectively. The explicit expressions of all the above given fundamental solutions can be found in Appendix A. Comparing the explicit expressions for $\widehat{U}_{ij}^{s}$ and $\widehat{U}_{ij}^{ss}$ in Appendix A it is obvious that both fundamental solutions are identical.

Incompressible model. In Section 2, the governing equations for the u_i^s–p formulation were obtained by applying a limit to R and setting $\alpha = 1$. Unfortunately, this limit can be applied to the fundamental solutions of the compressible case only in 3-d. In 2-d, this limit is not finite. So, these solutions must be calculated independently where the procedure is the same as before. First, the determinants with their respective roots are calculated. However, here, both formulations, the u_i^s–p formulation and the u_i^s–u_i^f formulation, have different roots indicating that two different incompressible models are considered as discussed in Section 2.

First, the u_i^s–p formulation is discussed. In this representation, the determinants are

$$\text{2-d}: \qquad \det(\mathbf{B4}^{\text{incomp}}) = \frac{G\beta}{s\rho_f}\left(K + \frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)\left(\nabla^2 - s^2\lambda_1^2\right)\nabla^2, \tag{39}$$

$$\text{3-d}: \qquad \det(\mathbf{B4}^{\text{incomp}}) = \frac{G^2\beta}{s\rho_f}\left(K + \frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)^2\left(\nabla^2 - s^2\lambda_1^2\right)\nabla^2, \tag{40}$$

with the roots

$$\lambda_1^2 = \frac{\rho + \rho_f\left(\frac{1}{\beta} - 2\right)}{K + \frac{4}{3}G}, \quad \lambda_3^2 = \frac{\rho - \beta\rho_f}{G}. \tag{41}$$

This yields an operator equation similar to (33),

$$\left(\nabla^2 - s^2\lambda_3^2\right)\left(\nabla^2 - s^2\lambda_1^2\right)\nabla^2\psi + \delta(\mathbf{x} - \mathbf{y}) = 0, \tag{42}$$

using the appropriate abbreviation for ψ corresponding to (34). Due to the Laplacian operator in (42) this is no longer an iterated modified Helmholtz equation but can be solved in a similar way by splitting the operator in Helmholtz and Laplace equations. The solution is

$$\text{2-d}: \qquad \psi = \frac{1}{2\pi s^4}\left[\frac{K_0(\lambda_1 sr)}{\left(\lambda_1^2 - \lambda_3^2\right)\lambda_1^2} - \frac{\ln(r)}{\lambda_1^2\lambda_3^2} + \frac{K_0(\lambda_3 sr)}{\left(\lambda_3^2 - \lambda_1^2\right)\lambda_3^2}\right], \tag{43}$$

$$\text{3-d}: \qquad \psi = \frac{1}{4\pi r s^4}\left[\frac{e^{-\lambda_1 sr}}{\left(\lambda_1^2 - \lambda_2^2\right)\left(\lambda_1^2 - \lambda_3^2\right)} + \frac{1}{\lambda_1^2\lambda_3^2} + \frac{e^{-\lambda_3 sr}}{\lambda_3^2\left(\lambda_3^2 - \lambda_1^2\right)}\right]. \tag{44}$$

As remarked at the beginning of this subsection, in 3-d, the incompressible solutions (44) are the limit values of the compressible results (36) for $\lambda_2 \to 0$. Contrary, in 2-d, the compressible solutions (35) tend to infinity for $\lambda_2 \to 0$, i.e., to calculate the solution (43), Eq. (42) has to be solved.

For the u_i^s–u_i^f formulation the determinants are found to be

$$\text{2-d}: \qquad \det(\mathbf{B6}^{\text{incomp}}) = \frac{s^4 G\phi^4\rho_f^2}{\beta^2}\left(K + \frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)\left(\nabla^2 - s^2\lambda_1^2\right), \tag{45}$$

$$\text{3-d}: \qquad \det(\mathbf{B6}^{\text{incomp}}) = \frac{-s^6 G^2 \phi^6 \rho_f^3}{\beta^3}\left(K + \frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)^2\left(\nabla^2 - s^2\lambda_1^2\right), \tag{46}$$

with the roots

$$\lambda_1^2 = \frac{\rho - \beta\rho_f}{K + \frac{4}{3}G}, \quad \lambda_3^2 = \frac{\rho - \beta\rho_f}{G}. \tag{47}$$

These determinants yield a modified iterated Helmholtz operator as governing equation similar to (33),

$$\left(\nabla^2 - s^2\lambda_3^2\right)\left(\nabla^2 - s^2\lambda_1^2\right)\psi + \delta(\mathbf{x} - \mathbf{y}) = 0, \tag{48}$$

using the proper abbreviation

$$\begin{aligned} \text{2-d}: &\qquad \mathbf{B6}^{\text{incomp}} : \psi = G\frac{s^4\phi^4\rho_f^2}{\beta^2}\left(K + \frac{4}{3}G\right)\varphi, \\ \text{3-d}: &\qquad \mathbf{B6}^{\text{incomp}} : \psi = -G^2\frac{s^6\phi^6\rho_f^3}{\beta^3}\left(K + \frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)\varphi. \end{aligned} \tag{49}$$

The solution of the modified Helmholtz equation (48) is

$$\text{2-d}: \qquad \psi = \frac{1}{2\pi s^2}\frac{1}{\lambda_3^2 - \lambda_1^2}\left[K_0(\lambda_3 sr) - K_0(\lambda_1 sr)\right], \tag{50}$$

$$\text{3-d}: \qquad \psi = \frac{1}{4\pi r s^2}\frac{1}{\lambda_3^2 - \lambda_1^2}\left[e^{-\lambda_3 sr} - e^{-\lambda_1 sr}\right]. \tag{51}$$

These solutions essentially differ from the corresponding ones in the u_i^s–p formulation (43) and (44). The terms $\ln(r)/(\lambda_1^2\lambda_3^2)$ or $1/(\lambda_1^2\lambda_3^2)$ produced by the limit $\lambda_2 \to 0$ in (43) and (44) are no longer present. So, obviously this simplified incompressible model will produce different results compared to the incompressible u_i^s–p formulation.

Concerning the waves represented in both models the following observations are made. In both formulations, the third root λ_3 corresponding to the shear wave velocity is not changed because incompressibility can only affect volumetric changes. Contrary, the compressional waves have to change as observed by the vanishing root λ_2 and the different root λ_1. Here, also the difference between both formulations is obvious. In the u_i^s–p formulation the smaller value λ_2, corresponding to the faster compression wave, goes to zero. The larger value λ_1, corresponding to the slower compressional wave, survives. Reflecting the physics behind these two compressional waves this behavior is explainable. In case of the fast compressional wave, the solid and the fluid move in phase. If the solid material is assumed to be incompressible it has no longer any volumetric deformation and, subsequently, the wave speed tends to infinity respective the corresponding λ_2 to zero. In case of the slow compressional wave, the solid and fluid move in opposite phase. This relative movement is still possible if the solid material is incompressible.

These physical considerations are well represented in the u_i^s–p formulation. Contrary, in the u_i^s–u_i^f formulation, no root λ_2 exists, i.e., the determinants (45) or (46) are only of second or third order in ∇^2 in 2-d or 3-d, respectively. This reflects the fact that this incompressible model is not achieved by a limit as in the u_i^s–p formulation. Only from physics it can be concluded that the fast compressional wave vanishes, however, the surviving wave has a different wave velocity compared to the other formulation.

Finally, the incompressible fundamental solutions are found for the u_i^s–p formulation:

$$\mathbf{G4}^{\text{incomp}} = \frac{s\rho_f}{G\beta\left(K + \frac{4}{3}G\right)}\begin{bmatrix} (F^*\nabla^2 + AD^*)\delta_{ij} - F^*\partial_{ij} & -A(1-\beta)s\partial_i \\ -A(1-\beta)\partial_i & A\left(\left(K + \frac{1}{3}G\right)\nabla^2 + A\right) \end{bmatrix}\psi \tag{52}$$

with the function ψ taken from (43) in the 2-d case or from (44) in the 3-d case. Different from the compressible case, here, the constants are $D^* = \beta/(s\rho_f)\nabla^2$ and $F^* = (K + 1/3G)D^* - (1-\beta)^2 s$. For the u_i^s–u_i^f formulation the matrix of fundamental solutions is

$$\mathbf{G6}^{\text{incomp}} = \begin{bmatrix} \widehat{U}_{ij}^{ss} & \widehat{U}_{ij}^{sf} \\ \widehat{U}_{ij}^{fs} & \widehat{U}_{ij}^{ff} \end{bmatrix} = \frac{\beta^2\left(K+\frac{1}{3}G\right)}{Gs^4\phi^4\rho_f^2\left(K+\frac{4}{3}G\right)} \begin{bmatrix} (M_0E + E^2\nabla^2)\delta_{ij} - E^2\partial_{ij} & -C(M_0 + E\nabla^2)\delta_{ij} + CE\partial_{ij} \\ -C(M_0 + E\nabla^2)\delta_{ij} + CE\partial_{ij} & B(M_0 + \nabla^2 E)\delta_{ij} - C^2\partial_{ij} \end{bmatrix}\psi \tag{53}$$

with the abbreviations B, C and E from the compressible case and

$$M_0 = \frac{s^2\rho_f\phi^2 G}{\beta\left(K+\frac{1}{3}G\right)}\left(s^2\lambda_3^2 - \nabla^2\right).$$

In Eq. (53), the function ψ has to be taken from (50) in the 2-d case or from (51) in the 3-d case. The final result can be summarized in the following form:

$$\mathbf{G6}^{\text{incomp}} = \begin{bmatrix} 1 & \dfrac{\phi-\beta}{\phi} \\ \dfrac{\phi-\beta}{\phi} & \dfrac{(\phi-\beta)^2}{\phi^2} \end{bmatrix} \widehat{U}_{ij}^{ss}. \tag{54}$$

The explicit expression of $\widehat{U}_{ij}^{ss}$ is given in Appendix A. The solution (54) makes it obvious that the underlying model for incompressibility is not sufficient because this result can be interpreted as totally dominant solid displacements, i.e., the fluid influences only the material data of the bulk material but not the behavior. This seems to be a very crude approximation of the realistic behavior, especially under the aspect of wave propagation.

In general, all the above derived incompressible solutions show that the assumption of incompressible constituents yields an infinite wave speed of the fast compressional wave. Contrary, if only one constituent is assumed to be incompressible all wave types still have finite wave speeds. It was also shown that in the incompressible model of the u_i^s–u_i^f formulation one compressional wave disappears. This makes in the authors' opinion no sense. However, the other model for incompressibility used in the u_i^s–p formulation, i.e., $R \to \infty$ and $\alpha = 1$, which cannot be introduced to the constitutive equations of the partial stress formulation, as discussed in Section 2, can be inserted into the final compressible fundamental solutions (A.5) and (A.6) of the u_i^s–u_i^f formulation. Due to the different model assumptions such incompressible fundamental solutions for the u_i^s–u_i^f formulation are different from (54).

4. Singular behavior

The singular behavior of the above given fundamental solutions can be found by a series expansion with respect to the variable r. This variable is found in these solutions either in the exponential function in the 3-d solutions or in the Bessel functions in case of 2-d. Else, only powers of r appear. So, it is sufficient to insert in the fundamental solutions (A.1)–(A.6) the following series expansions:

$$e^{-\lambda_k sr} = \sum_{\ell=0}^{\infty} \frac{(-\lambda_k sr)^\ell}{\ell!} = 1 - \lambda_k sr + \lambda_k^2 s^2 r^2 + \mathcal{O}(r^3) \tag{55}$$

for the exponential function, and

$$K_0(\lambda_k sr) = -(\ln(\lambda_k sr) - \ln 2 + \gamma) + \mathcal{O}(r^2), \tag{56a}$$

$$K_1(\lambda_k sr) = \frac{1}{\lambda_k sr} + \frac{\lambda_k sr}{2}\left(\ln(\lambda_k sr) - \ln 2 + \gamma - \frac{1}{2}\right) + \mathcal{O}(r^3),$$
$$\gamma = \lim_{n\to\infty}\left(\sum_{v=1}^{n}\frac{1}{v} - \ln n\right) = 0.577216 \text{ (Euler-constant)} \tag{56b}$$

for the Bessel functions. Inserting these series in the fundamental solutions and a subsequent ordering with respect to the power of r yields the singular behavior.

4.1. u_i^s–p formulation

For the u_i^s–p formulation the compressible as well as the incompressible solution behaves equal. In 3-d, it is found

$$\hat{P}_i^s, \hat{U}_i^f = \mathcal{O}(r^0), \tag{57a}$$

$$\hat{U}_{ij}^s = \underbrace{\frac{1}{16\pi G(1-v)}\{r_{,i}r_{,j} + (3-4v)\delta_{ij}\}\frac{1}{r}}_{\text{elastostatic fundamental solution}} + \mathcal{O}(r^0), \tag{57b}$$

$$\hat{P}^f = \frac{\rho_f s}{4\pi\beta}\frac{1}{r} + \mathcal{O}(r^0), \tag{57c}$$

and in 2-d a similar result is achieved

$$\hat{P}_i^s, \hat{U}_i^f = \mathcal{O}(r^0), \tag{58a}$$

$$\hat{U}_{ij}^s = \underbrace{\frac{1}{8\pi G(1-v)}\{r_{,i}r_{,j} - (3-4v)\delta_{ij}\ln r\}}_{\text{elastostatic fundamental solution}} + \mathcal{O}(r^0), \tag{58b}$$

$$\hat{P}^f = \frac{-\rho_f s}{2\pi\beta}\ln r + \mathcal{O}(r^0). \tag{58c}$$

So, the singular behavior is the same as in elastostatics or acoustics, i.e., the poroelastic fundamental solutions are only weakly singular or even regular. Again, note that there is no different behavior between the compressible or incompressible model.

4.2. u_i^s–u_i^f formulation

For the u_i^s–u_i^f formulation, the singular behavior is different from the above discussed formulation. This is not surprising because looking at the differential operator **B6** (26) it is observed that this operator has in the lower part of the main diagonal no Laplacian operator contrary to the operator **B4** (24). This is also represented in the fact that the three members $\hat{U}_{ij}^{fs}$, $\hat{U}_{ij}^{sf}$, and $\hat{U}_{ij}^{ff}$ of **G6** are composed by the fundamental solution $\hat{U}_{ij}^{ss}$ and some additional term. In detail, the following singularities are found for the 3-d case:

$$\widehat{U}_{ij}^{\mathrm{ss}} = \underbrace{\frac{1}{16\pi G(1-\nu)}\left\{r_{,i}r_{,j} + (3-4\nu)\delta_{ij}\right\}\frac{1}{r}}_{\text{elastostatic fundamental solution}} + \mathcal{O}(r^0), \tag{59a}$$

$$\widehat{U}_{ij}^{\mathrm{sf}} = \widehat{U}_{ij}^{\mathrm{fs}} = \frac{1}{16\pi G(1-\nu)}\left\{\frac{\phi-\beta}{\phi}(r_{,i}r_{,j} + (3-4\nu)\delta_{ij}) + \frac{\alpha-\beta}{\phi}(r_{,i}r_{,j} - \delta_{ij})(1-2\nu)\right\}\frac{1}{r} + \mathcal{O}(r^0), \tag{59b}$$

$$\widehat{U}_{ij}^{\mathrm{ff}} = \frac{\beta}{4\pi\phi^2 s^2 \rho_{\mathrm{f}}}\left\{3r_{,i}r_{,j} - \delta_{ij}\right\}\frac{1}{r^3} + \mathcal{O}(r^{-1}) \tag{59c}$$

and for the 2-d case

$$\widehat{U}_{ij}^{\mathrm{ss}} = \underbrace{\frac{1}{8\pi G(1-\nu)}\left\{r_{,i}r_{,j} - (3-4\nu)\delta_{ij}\ln r\right\}}_{\text{elastostatic fundamental solution}} + \mathcal{O}(r^0), \tag{60a}$$

$$\begin{aligned}\widehat{U}_{ij}^{\mathrm{sf}} &= \widehat{U}_{ij}^{\mathrm{fs}} \\ &= \frac{1}{8\pi G(1-\nu)}\left\{\frac{\phi-\beta}{\phi}(r_{,i}r_{,j} - (3-4\nu)\delta_{ij}\ln r) + \frac{\alpha-\beta}{\phi}(r_{,i}r_{,j} + \delta_{ij}\ln r)(1-2\nu)\right\} + \mathcal{O}(r^0),\end{aligned} \tag{60b}$$

$$\widehat{U}_{ij}^{\mathrm{ff}} = \frac{\beta}{2\pi\phi^2 s^2 \rho_{\mathrm{f}}}\left\{2r_{,i}r_{,j} - \delta_{ij}\right\}\frac{1}{r^2} + \mathcal{O}(\ln r). \tag{60c}$$

In Eqs. (59c) and (60c), it becomes obvious that these solutions are hyper-singular, whereas all other solutions are weakly singular. Also, in (59b) and (60b) the elastostatic singularity of (59a) and (60a), respectively is identified with some additional poroelastic terms.

In case of the incompressible model, clearly, due to the connected form of (54) all four fundamental solutions have the same order of singularity namely that of $\widehat{U}_{ij}^{\mathrm{ss}}$. The limit of this solution (A.7) or (A.8) yields as in the compressible case the elastostatic fundamental solution. However, no hyper-singular behavior exists for the incompressible solutions.

5. Visualization of some fundamental solutions

Finally, some exemplary fundamental solutions are calculated to visualize the principal behavior and the difference between the compressible and incompressible model. Despite the differences in both incompressible models, i.e., in the u_i^{s}–p formulation and in the u_i^{s}–u_i^{f} formulation, the principal effects which can be visualized are similar. Therefore, next, only the visualization for the u_i^{s}–p formulation and for this formulation only the displacement due to a point force $\widehat{U}_{ij}^{\mathrm{s}}$ and the pressure due to a source $\widehat{P}^{\mathrm{f}}$ in 3-d are presented.

Exemplary for a material which can be modeled incompressible as well as compressible a soil is chosen. The material data (see Table 1) are taken from literature (Kim and Kingsbury, 1979). The incompressibility condition (5) yields for this material:

$$\frac{K}{K^{\mathrm{s}}} = 0.019, \quad \frac{K}{K^{\mathrm{f}}} = 0.0636. \tag{61}$$

So, it can be expected that the fundamental solutions of the compressible and incompressible model show a similar behavior.

Table 1
Material data of a soil (coarse sand)

	K (N/m^2)	G (N/m^2)	ρ (kg/m^3)	ϕ	K^s (N/m^2)	ρ_f (kg/m^3)	K^f (N/m^2)	κ (m^4/N s)
Soil	2.1×10^8	9.8×10^7	1884	0.48	1.1×10^{10}	1000	3.3×10^9	3.55×10^{-9}

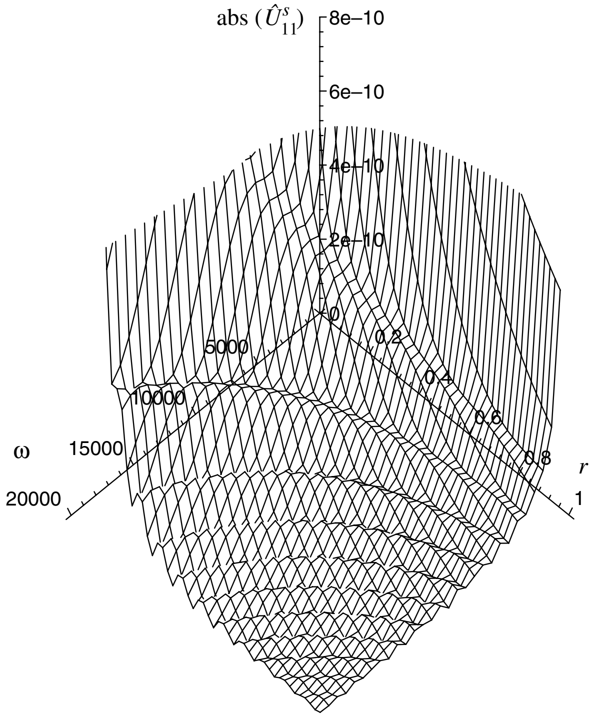

Fig. 1. Displacement fundamental solution abs$\left(\widehat{U}^s_{11}\right)$ versus frequency, ω and distance, r.

First, in Fig. 1, the displacement fundamental solution abs$\left(\widehat{U}^s_{11}\right)$ is depicted versus the distance r and the frequency ω. To introduce in the fundamental solutions from Appendix A the frequency instead of the complex Laplace variable s, simply the real part of s is set to zero, i.e., $s = i\omega$. Further, the absolute value of the complex valued displacement solution, i.e., the amplitude, is given in Fig. 1 and the range of values is restricted at the singularity. The singular behavior for small values of r is nearly independent of the frequency. Away from the origin the solution shows a wave like form with smaller amplitudes for higher frequencies.

In the following, to have a better insight into the behavior of the fundamental solutions, the distance r is kept constant and the frequency is varied. Further, all results, i.e., the displacement and pressure results are normalized to their singular behavior (57b) and (57c), respectively. Additionally to the frequency results also the time-dependent fundamental solutions are calculated by an inverse Laplace transform. However, not the impulse response functions are presented but the response due to a Heaviside (unit step function) time history of the load. This is achieved by the convolution between the fundamental solution and the Heaviside function. Both operations, the inverse transform and the convolution, are performed within one calculation using the Convolution Quadrature Method proposed by Lubich (1988).

In Fig. 2, the normalized displacement fundamental solution $\mathrm{abs}\left(\widehat{U}_{11}^{s}\right)$ is plotted versus frequency for the compressible and the incompressible model. This study is given for two points at $r = 0.1$ m and at $r = 0.5$ m distance from the origin. For moderate frequencies and small r both solutions, compressible and incompressible, are very similar whereas for higher frequencies differences are observed. This is in accordance with the model. The fast compressional wave which speed tends to infinity influences only the short time behavior, i.e., the higher frequencies. Hence, if this wave vanishes only the high frequency range of the

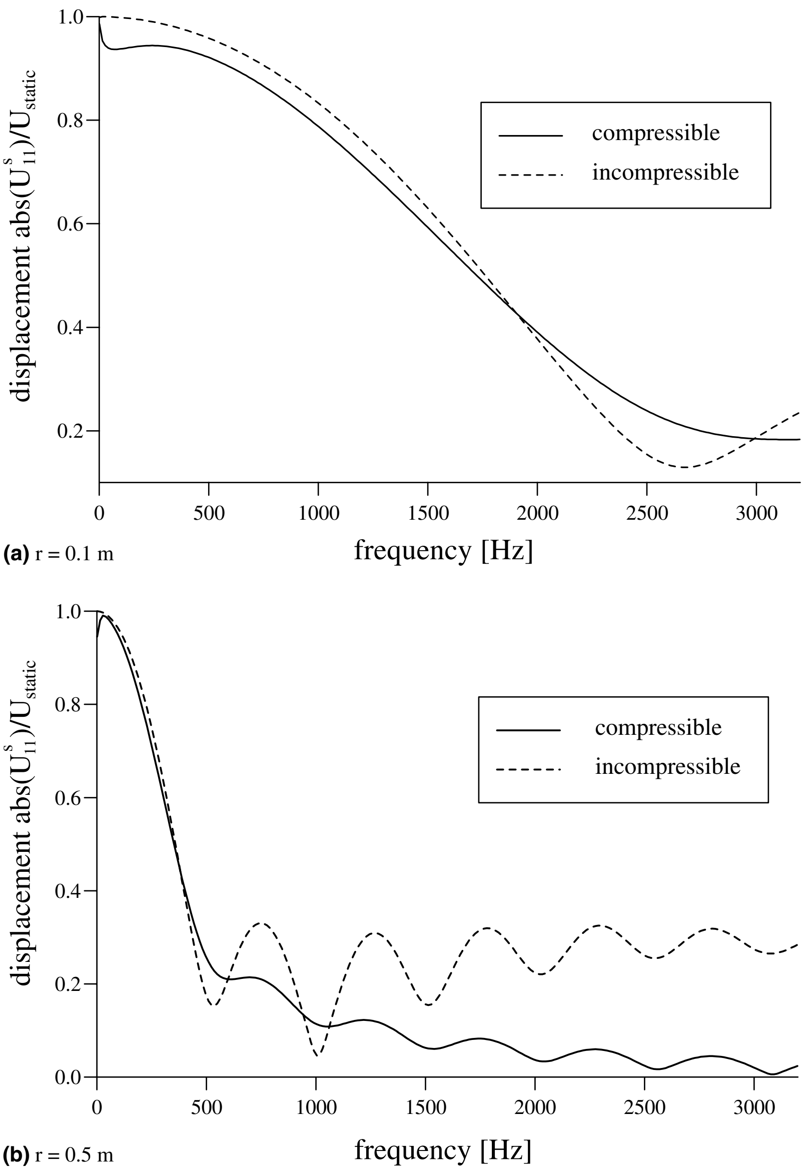

Fig. 2. Displacement fundamental solution $\mathrm{abs}\left(\widehat{U}_{11}^{s}\right)$ normalized with U_{static} (57b) versus frequency ω: Comparison compressible and incompressible model at different distances r.

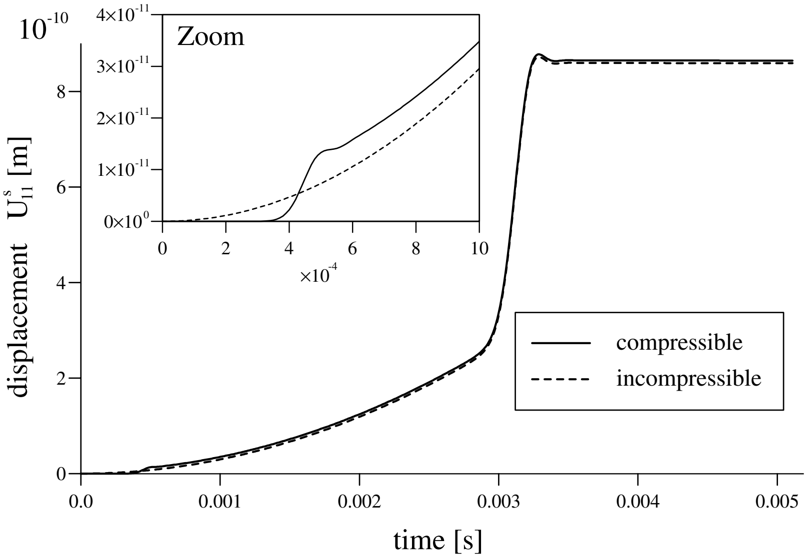

Fig. 3. Displacement unit step response function U_{11}^{s} versus time t: Comparison of the compressible and incompressible model at a distance $r = 0.5$ m.

solution is affected. The singular behavior, i.e., the limit $\omega \rightarrow 0$, is identical for the compressible and incompressible solution. However, for small but nonzero frequencies the solutions differ for the two models, which is well visible for $r = 0.1$ m. This difference may be explained with the change in the speed of the slow compressional wave.

Except of the last effect all these differences of the compressible and the incompressible model are also visible in time domain. Therefore, in Fig. 3 the time dependent displacement response due to a Heaviside load in time is depicted versus time at a distance $r = 0.5$ m. There are, as expected, not too much differences visible in the long time behavior. The two jumps in the graph at $t = 0.0004$ s and at $t = 0.0031$ s correspond to the fast compressional wave and to the shear wave, respectively. In the zoom, it becomes visible that in the incompressible model (dashed line) the compressional wave speed tends to infinity, i.e., the arrival time tends to zero. Else, this time dependent plot of the fundamental solution shows that for this material the incompressible model can be chosen if not the early time response is under consideration. However, it must be remarked that for other material data, especially if they violate the incompressibility condition (5), both models show large differences over the complete observation period.

Next, in Fig. 4, the normalized pressure due to a source in the fluid is considered. For this solution the largest differences are expected because the pore pressure is no longer a free variable in the incompressible model. Further, in an incompressible fluid a change in pressure is immediate at every point r, hence the pressure cannot show a strong time, respectively frequency, dependence. These effects are observed in Fig. 4, where both results differ by several decades in the absolute values. The compressible pressure is much smaller than the solution in the incompressible model and shows a more pronounced frequency dependence. However, for very small frequency, i.e., for the long time behavior, both solutions tend to the same value. It should be remarked that in Fig. 4 a logarithmic scale for the pressure is used which on the one hand enables this representation at all but on the other hand distorts the frequency dependence.

In the time domain these considerations are confirmed. In Fig. 5, the pressure due to a Heaviside time history of the load is depicted versus time at a distance $r = 0.5$ m. Note, in Fig. 5 a different time scale

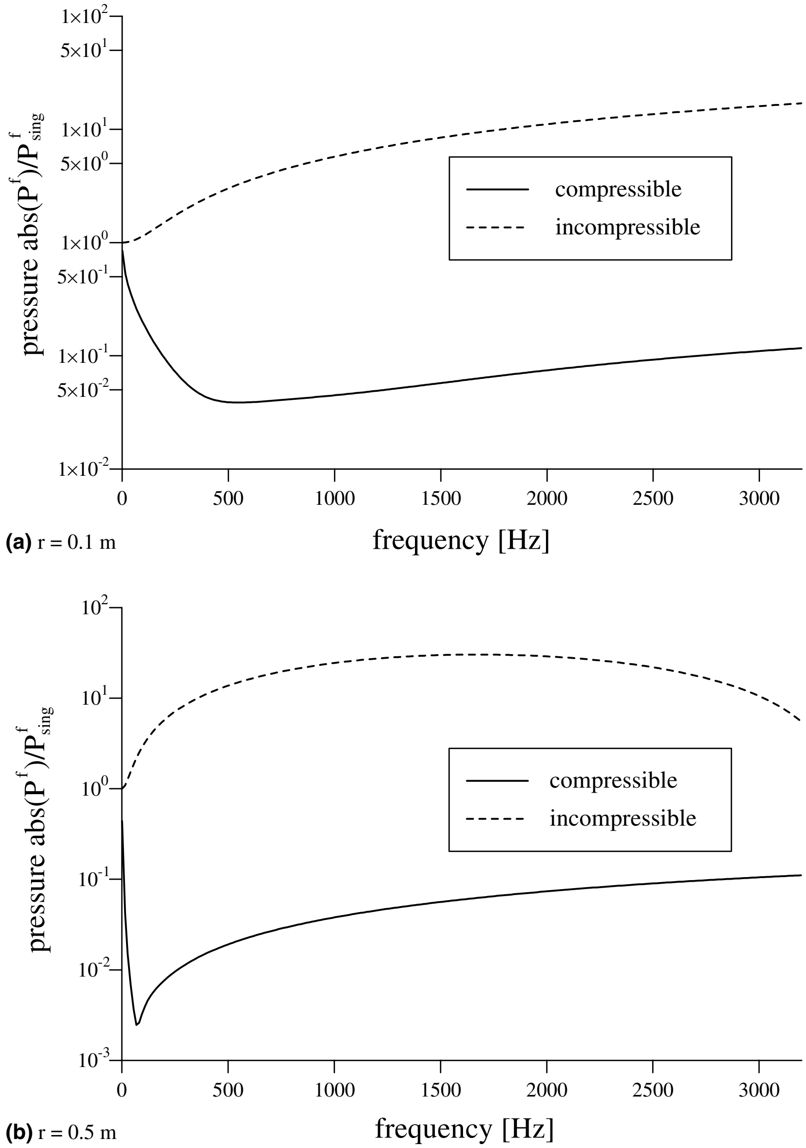

Fig. 4. Pressure fundamental solution $\mathrm{abs}\left(\widehat{P}^{\mathrm{f}}\right)$ normalized with $\widehat{P}^{\mathrm{f}}_{\mathrm{sing}}$ (57c) versus frequency, ω. Comparison: compressible and incompressible model at different distances r.

compared to Fig. 3 is used. The pressure is mostly zero with the exception of the arrival time of the compressional wave at $t = 0.0004$ s. There, in the compressible solution an impulse is visible. The same impulse is also visible in the incompressible solution, however, at $t = 0$ s. Further, the characteristics of this shock wave is different for both models. In the more stiff incompressible model a more pronounced and larger impulse has been calculated compared to the compressible model. Naturally, the amplitude and sharpness of such a shock wave is dependent on the time discretization used and other parameters of the inverse transformation. However, in the comparison above for both the same parameters have been applied, so the results are comparable.

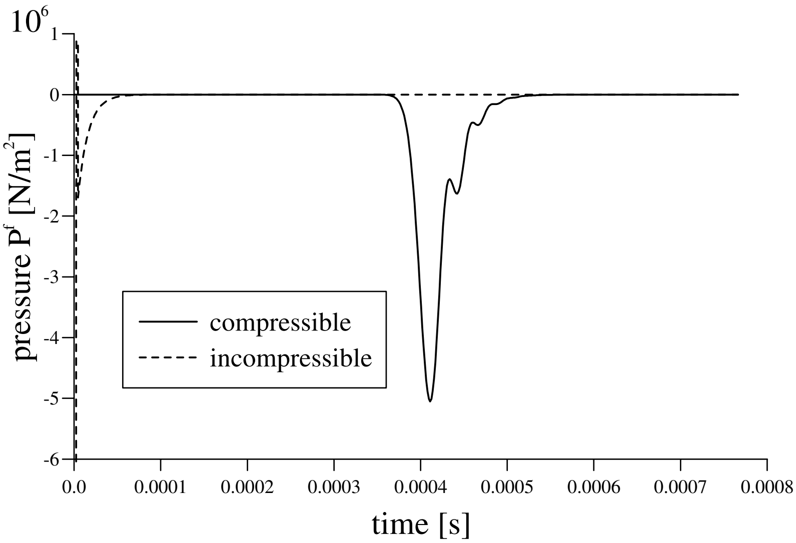

Fig. 5. Pressure unit step response function P^f versus time t: Comparison of the compressible and incompressible model at a distance $r = 0.5$ m.

6. Conclusions

Based on Biot's theory, in the present work, fundamental solutions for the special case of incompressible constituents are deduced and compared to the fundamental solutions of the compressible case. This has been done not only for the representation with the solid displacement and the pore pressure as unknowns, but also for the solid displacement and fluid displacement formulation. For both representations different models for incompressible constituents are given. The fundamental solutions are determined using the method of Hörmander.

The derivation of the fundamental solutions has confirmed the known fact that the solid displacement and the pore pressure are sufficient to describe the behavior of a poroelastic continuum. Further, it has been shown that the incompressible model of the solid and fluid displacement formulation is not suitable to describe the dynamic behavior of a poroelastic medium. In general, an incompressible model assumes an infinite wave speed of the fast compressional wave, i.e., this wave form is neglected. Hence, the question arise whether such an approximation makes sense in a wave propagation calculation. The presented fundamental solutions show differences for higher frequencies, i.e., short times, in comparison to the compressible model. Therefore, it can be concluded that an incompressible model can only be used in wave propagation problems if not the short time behavior is considered and also if the ratios of the compression moduli are very small.

Appendix A. Explicit expressions for the fundamental solutions

The explicit expressions of the poroelastodynamic fundamental solutions for the unknowns solid displacement, u_i^s, and pore pressure, p, and for solid displacement and fluid displacement, u_i^s and u_i^f, are given in the following for a 2-d and a 3-d continuum, for compressible as well as incompressible constituents.

A.1. Solid displacement, u_i^s and pore pressure, p

A.1.1. Compressible model

3-d. The elements of the matrix $\mathbf{G4}^{\mathrm{comp}}$ (37) are the displacements caused by a Dirac force in the solid

$$\widehat{U}_{ij}^{s} = \frac{1}{4\pi r(\rho - \beta\rho_f)s^2}\left[R_1\frac{\lambda_4^2-\lambda_2^2}{\lambda_1^2-\lambda_2^2}\mathrm{e}^{-\lambda_1 sr} - R_2\frac{\lambda_4^2-\lambda_1^2}{\lambda_1^2-\lambda_2^2}\mathrm{e}^{-\lambda_2 sr} + \left(\delta_{ij}\lambda_3^2 s^2 - R_3\right)\mathrm{e}^{-\lambda_3 sr}\right] \tag{A.1a}$$

with $R_k = (3r_{,i}r_{,j} - \delta_{ij})/r^2 + \lambda_k s(3r_{,i}r_{,j} - \delta_{ij})/r + \lambda_k^2 s^2 r_{,i}r_{,j}$ and $\lambda_4^2 = (\rho - \beta\rho_f)/(K + 4/3G)$. The pressure caused by the same load is

$$\widehat{P}_j^{s} = \frac{(\alpha-\beta)\rho_f r_{,j}}{4\pi\beta s\left(K+\frac{4}{3}G\right)r\left(\lambda_1^2-\lambda_2^2\right)}\left[\left(\lambda_1 s + \frac{1}{r}\right)\mathrm{e}^{-\lambda_1 sr} - \left(\lambda_2 s + \frac{1}{r}\right)\mathrm{e}^{-\lambda_2 sr}\right]. \tag{A.1b}$$

For a Dirac source in the fluid the respective displacement solution is

$$\widehat{U}_i^{f} = s\widehat{P}_i^{s} \tag{A.1c}$$

and the pressure

$$\widehat{P}^{f} = \frac{s\rho_f}{4\pi r\beta\left(\lambda_1^2-\lambda_2^2\right)}\left[\left(\lambda_1^2-\lambda_4^2\right)\mathrm{e}^{-\lambda_1 sr} - \left(\lambda_2^2-\lambda_4^2\right)\mathrm{e}^{-\lambda_2 sr}\right]. \tag{A.1d}$$

In the above given solutions, the roots λ_i, $i = 1, 2, 3$ from (32) are used.

2-d. In 2-d, the expressions for displacements induced by a force in the solid are

$$\widehat{U}_{ij}^{s} = \frac{1}{2\pi s^2(\rho-\beta\rho_f)}\left[\frac{\lambda_4^2-\lambda_2^2}{\lambda_1^2-\lambda_2^2}R_1^{2d} - \frac{\lambda_4^2-\lambda_1^2}{\lambda_1^2-\lambda_2^2}R_2^{2d} - R_3^{2d} + \delta_{ij}s^2\lambda_3^2 K_0(\lambda_3 sr)\right] \tag{A.2a}$$

and the pressure for the same load is

$$\widehat{P}_j^{s} = \frac{\rho_f(\alpha-\beta)r_{,i}}{2\pi\beta}\,\frac{K_1(\lambda_1 sr)\lambda_1 - K_1(\lambda_2 sr)\lambda_2}{\left(\lambda_1^2-\lambda_2^2\right)\left(K+\frac{4}{3}G\right)}. \tag{A.2b}$$

The roots λ_i, $i = 1, 2, 3$ are the same as in the 3-d case (32). The displacement fundamental solution for a source in the fluid is

$$\widehat{U}_i^{f} = s\widehat{P}_j^{s} \tag{A.2c}$$

and the pressure solution is

$$\widehat{P}^{f} = \frac{s\rho_f}{2\pi\beta}\,\frac{\left(\lambda_1^2-\lambda_4^2\right)K_0(\lambda_1 sr) - \left(\lambda_2^2-\lambda_4^2\right)K_0(\lambda_2 sr)}{\lambda_1^2-\lambda_2^2}. \tag{A.2d}$$

The abbreviation

$$R_k^{2d} = \frac{2r_{,i}r_{,j} - \delta_{ij}}{r}\lambda_k s K_1(\lambda_k sr) + r_{,i}r_{,j}s^2\lambda_k^2 K_0(\lambda_k sr)$$

is used in Eqs. (A.2). Further, K_0 and K_1 denote the modified Bessel functions of second kind.

A.1.2. Incompressible model

3-d. For the case of incompressible constituents, the displacements caused by a Dirac force in the solid are

$$\widehat{U}_{ij}^{\mathrm{s}} = \frac{1}{4\pi r(\rho - \beta\rho_{\mathrm{f}})s^2}\left[R_1\frac{\lambda_4^2}{\lambda_1^2}\mathrm{e}^{-\lambda_1 sr} - R_2\frac{\lambda_4^2-\lambda_1^2}{\lambda_1^2} + \left(\delta_{ij}\lambda_3^2 s^2 - R_3\right)\mathrm{e}^{-\lambda_3 sr}\right] \tag{A.3a}$$

with same abbreviations (R_1, R_2, λ_4) as in the compressible case and $\lambda_{1,3}$ from (41). The pressure caused by the same load is

$$\widehat{P}_j^{\mathrm{s}} = \frac{(\alpha-\beta)\rho_{\mathrm{f}} r_{,j}}{4\pi\beta\left(K+\frac{4}{3}G\right)rs\lambda_1^2}\left[\left(\lambda_1 s + \frac{1}{r}\right)\mathrm{e}^{-\lambda_1 sr} - \frac{1}{r}\right]. \tag{A.3b}$$

For a Dirac source in the fluid the respective displacement solution is

$$\widehat{U}_i^{\mathrm{f}} = s\widehat{P}_i^{\mathrm{s}} \tag{A.3c}$$

and the pressure solution

$$\widehat{P}^{\mathrm{f}} = \frac{s\rho_{\mathrm{f}}}{4\pi r\beta\lambda_1^2}\left[\left(\lambda_1^2-\lambda_4^2\right)\mathrm{e}^{-\lambda_1 sr} + \lambda_4^2\right]. \tag{A.3d}$$

2-d. The above presented 3-d solution for the incompressible model can be simply achieved by the limit $\lambda_2 \to 0$, contrary to the 2-d solutions as shown in Section 3. Computing them following the formulas in Section 3 yields for the displacement fundamental solutions

$$\widehat{U}_{ij}^{\mathrm{s}} = \frac{1}{2\pi s^2(\rho - \beta\rho_{\mathrm{f}})}\left[\frac{\lambda_4^2}{\lambda_1^2}R_1^{2d} - \frac{\lambda_4^2-\lambda_1^2}{\lambda_1^2}\frac{2r_{,i}r_{,j}-\delta_{ij}}{r^2} - R_3^{2d} + \delta_{ij}s^2\lambda_3^2 K_0(\lambda_3 sr)\right] \tag{A.4a}$$

with the roots λ_1 and λ_3 from Eq. (41) and the other abbreviations from the compressible solution. Eq. (A.4a) is the result due to a single force in the solid. The respective pressure solution for such a load is

$$\widehat{P}_j^{\mathrm{s}} = \frac{r_{,i}\rho_{\mathrm{f}}}{2\pi s\beta}\frac{(1-\beta)(\lambda_1 srK_1(\lambda_1 sr) - 1)}{\lambda_1^2 r\left(K+\frac{4}{3}G\right)}. \tag{A.4b}$$

The result due to a source in the fluid is given by

$$\widehat{U}_i^{\mathrm{f}} = s\widehat{P}_j^{\mathrm{s}} \tag{A.4c}$$

and the pressure by

$$\widehat{P}^{\mathrm{f}} = \frac{s\rho_{\mathrm{f}}}{2\pi\beta}\frac{\left(\lambda_1^2-\lambda_4^2\right)K_0(\lambda_1 sr) - \lambda_4^2\ln(r)}{\lambda_1^2}. \tag{A.4d}$$

A.2. Solid displacement, u_i^s and fluid displacement, u_i^f

A.2.1. Compressible model

3-d. The explicit expressions of the poroelastodynamic fundamental solutions are given in the following. The four elements of the matrix $\mathbf{G6}^{\mathrm{comp}}$ (38) are the displacements caused by a Dirac force in the solid

$$\widehat{U}_{ij}^{ss} = \frac{1}{4\pi r(\rho - \beta\rho_f)s^2}\left[R_1 \frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} e^{-\lambda_1 sr} - R_2 \frac{\lambda_4^2 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2} e^{-\lambda_2 sr} + \left(\delta_{ij}\lambda_3^2 s^2 - R_3\right) e^{-\lambda_3 sr}\right] \tag{A.5a}$$

with the roots λ_i, $i = 1, 2, 3$ from (32), $\lambda_4^2 = (\rho - \beta\rho_f)/(K + 4/3G)$, and the R_k from the u_i^s–p formulation. Comparing the above fundamental solution (A.5a) with the corresponding solution in the u_i^s–p formulation (A.1a) it is seen that they are identical. The relative fluid displacements caused by the same load and the solid displacements caused by a force in the fluid are

$$\widehat{U}_{ij}^{sf} = \widehat{U}_{ij}^{fs} = \frac{\phi - \beta}{\phi}\widehat{U}_{ij}^{ss} - \frac{1}{4\pi r s^2\left(K + \frac{4}{3}G\right)} \frac{\alpha - \beta}{\phi\left(\lambda_1^2 - \lambda_2^2\right)}\left\{R_1 e^{-\lambda_1 sr} - R_2 e^{-\lambda_2 sr}\right\}. \tag{A.5b}$$

For a Dirac force in the fluid the respective fluid displacement solution is

$$\begin{aligned}\widehat{U}_{ij}^{ff} = {} & \frac{(\phi - \beta)^2}{\phi^2}\widehat{U}_{ij}^{ss} + \frac{1}{4\pi r s^2(K + \frac{4}{3}G)} \frac{\beta}{\phi^2\rho_f(\lambda_1^2 - \lambda_2^2)}\left\{R_1 e^{-\lambda_1 sr}\left(\lambda_1^2\left(K + \frac{4}{3}G\right) - (\rho - \beta\rho_f)\right.\right. \\ & \left.\left. - 2\rho_f(\phi - \beta)\frac{\alpha - \beta}{\beta}\right) - R_2 e^{-\lambda_2 sr}\left(\lambda_2^2\left(K + \frac{4}{3}G\right) - (\rho - \beta\rho_f) - 2\rho_f(\phi - \beta)\frac{\alpha - \beta}{\beta}\right)\right\}.\end{aligned} \tag{A.5c}$$

2-d. The 2-d fundamental solutions for the u_i^s–u_i^f formulation have a similar structure as the above given 3-d solutions. The four elements of the matrix $\mathbf{G6}^{comp}$ (38) are the displacements caused by a Dirac force in the solid

$$\widehat{U}_{ij}^{ss} = \frac{1}{2\pi s^2(\rho - \beta\rho_f)}\left[\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} R_1^{2d} - \frac{\lambda_4^2 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2} R_2^{2d} - R_3^{2d} + \delta_{ij} s^2 \lambda_3^2 K_0(\lambda_3 sr)\right], \tag{A.6a}$$

with the roots λ_i, $i = 1, 2, 3$ from (32), $\lambda_4^2 = (\rho - \beta\rho_f)/(K + 4/3G)$, and the R_k^{2d} from the u_i^s–p formulation. As before in the 3-d case, the fundamental solution (A.6a) is identical to the corresponding one (A.2a) of the u_i^s–p formulation. The relative fluid displacements caused by the same load are identical to the solid displacements caused by a force in the fluid

$$\widehat{U}_{ij}^{sf} = \widehat{U}_{ij}^{fs} = \frac{\phi - \beta}{\phi}\widehat{U}_{ij}^{ss} - \frac{1}{2\pi s^2\left(K + \frac{4}{3}G\right)} \frac{\alpha - \beta}{\phi\left(\lambda_1^2 - \lambda_2^2\right)}\left\{R_1^{2d} - R_2^{2d}\right\}. \tag{A.6b}$$

For a Dirac force in the fluid the respective relative fluid displacement solution is

$$\begin{aligned}\widehat{U}_{ij}^{ff} = {} & \frac{(\phi - \beta)^2}{\phi^2}\widehat{U}_{ij}^{ss} + \frac{\beta}{2\pi s^2\left(K + \frac{4}{3}G\right)\phi^2\rho_f} \frac{1}{\left(\lambda_1^2 - \lambda_2^2\right)}\left\{R_1^{2d}\left(\lambda_1^2\left(K + \frac{4}{3}G\right) - (\rho - \beta\rho_f)\right.\right. \\ & \left.\left. - 2\rho_f(\phi - \beta)\frac{\alpha - \beta}{\beta}\right) - R_2^{2d}\left(\lambda_2^2\left(K + \frac{4}{3}G\right) - (\rho - \beta\rho_f) - 2\rho_f(\phi - \beta)\frac{\alpha - \beta}{\beta}\right)\right\}.\end{aligned} \tag{A.6c}$$

A.2.2. Incompressible model

In this case the matrix of fundamental solutions is given in (54), but the explicit expression of the displacement fundamental solution due to a single force in the solid must be given. It is in 3-d:

$$\widehat{U}_{ij}^{ss} = \frac{1}{4\pi r(\rho - \beta\rho_f)s^2}\left[R_1 e^{-\lambda_1 sr} + \left(\delta_{ij}\lambda_3^2 s^2 - R_3\right) e^{-\lambda_3 sr}\right] \tag{A.7}$$

and in 2-d:

$$\widehat{U}_{ij}^{\mathrm{ss}} = \frac{1}{2\pi s^2(\rho - \beta\rho_{\mathrm{f}})}\left[R_1^{2d} - R_3^{2d} + \delta_{ij}s^2\lambda_3^2 K_0(\lambda_3 sr)\right] \tag{A.8}$$

with the roots λ_i, $i = 1, 3$ from (47) and the R_k and R_k^{2d} from the u_i^{s}–p formulation.

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