

# NON-BACKTRACKING RANDOM WALKS AND COGROWTH OF GRAPHS

RONALD ORTNER AND WOLFGANG WOESS

ABSTRACT. Let  $X$  be a locally finite, connected graph without vertices of degree 1. Non-backtracking random walk moves at each step with equal probability to one of the “forward” neighbours of the actual state, i.e., it does not go back along the preceding edge to the preceding state. This is not a Markov chain, but can be turned into a Markov chain whose state space is the set of oriented edges of  $X$ . Thus we obtain for infinite  $X$  that the  $n$ -step non-backtracking transition probabilities tend to zero, and we can also compute their limit when  $X$  is finite. This provides a short proof of old results concerning cogrowth of groups, and makes the extension of that result to arbitrary regular graphs rigorous. Even when  $X$  is non-regular, but *small cycles are dense in  $X$* , we show that the graph  $X$  is non-amenable if and only if the non-backtracking  $n$ -step transition probabilities decay exponentially fast. This is a partial generalization of the cogrowth criterion for regular graphs which comprises the original cogrowth criterion for finitely generated groups of GRIGORCHUK and COHEN.

## 1. INTRODUCTION AND RESULTS

Let  $X$  be the vertex set of a locally finite, connected graph, possibly with multiple edges and loops. We write  $e(x, y)$  for the number of edges between the vertices  $x$  and  $y$ , if  $y \neq x$ , while  $e(x, x)$  is *twice* the number of loops at  $x$  (see §2.B for a discussion). The degree of a vertex  $x \in X$  is  $\deg(x) = \sum_y e(x, y)$ . We assume that  $\deg(x) \geq 2$  for all  $x \in X$ . *Non-backtracking (simple) random walk* (NBRW) is the following random process: at the beginning, the walker starts at some vertex  $x$  and chooses with equal probability one of the incident edges. He steps to the other end of that edge. At the later steps, the rule is the same, but the walker selects with equal probability only among those incident edges that are different from the one transversed at the previous step.

We write  $q^{(n)}(x, y)$  for the probability that the random walker, starting at vertex  $x$ , is at vertex  $y$  at the  $n$ -th step. Note that NBRW is *not* a Markov chain on  $X$ . The defining property of a Markov chain, that “the future depends only on the actual state and not on the past”, is violated, since the walker has to remember the edge along which he reached the actual state before moving on.

However, it is easy to turn NBRW into a Markov chain by changing the state space: with each edge, we associate two oppositely oriented edges  $e, \check{e}$  (with  $\check{\check{e}} = e$ ). We write  $e^-$  and  $e^+$  for the initial and terminal vertex of the edge  $e$ , so that  $(\check{e})^- = e^+$  and  $(\check{e})^+ = e^-$ .

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(Note in particular, that for each a priori unoriented loop we get two oriented ones!) We now consider NBRW as a *Markov* process whose new state space is the set  $E = E(X)$  of oriented edges, with transition matrix  $Q_E = (q_E(e, f))_{e, f \in E}$  given by

$$q_E(e, f) = \begin{cases} \frac{1}{\deg(e^+) - 1}, & \text{if } e \rightarrow f, \text{ that is, } f^- = e^+ \text{ and } f \neq \check{e}, \\ 0, & \text{otherwise.} \end{cases}$$

Then there is the following link between edge-NBRW and vertex-NBRW.

**(1.1) Lemma.** *For arbitrary vertices  $x, y \in X$ ,*

$$q^{(n)}(x, y) = \frac{1}{\deg(x)} \sum_{\substack{e, f \in E: \\ e^+ = x, f^+ = y}} q_E^{(n)}(e, f),$$

where  $q_E^{(n)}$  denotes the  $n$ -step transition probabilities, i.e., the elements of the matrix power  $Q_E^n$ , with  $Q_E^0 = I_E$ , the identity matrix over  $E$ .

(Attention:  $q^{(n)}(x, y)$  is *not* the  $(x, y)$ -element of an  $n$ -th matrix power over  $X$ !)

The following result is then a consequence of basic Markov chain theory.

**(1.2) Theorem.** (a) *If  $X$  is finite, connected, with minimum degree 2, then for all  $x, y \in X$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left( q^{(1)}(x, y) + q^{(2)}(x, y) + \cdots + q^{(n)}(x, y) \right) = \frac{\deg(y)}{|E(X)|}.$$

(b) *If in addition to the assumptions of (a),  $X$  has minimum degree 3, then for all  $x, y \in X$ ,*

$$\begin{aligned} \lim_{n \rightarrow \infty} q^{(n)}(x, y) &= \frac{\deg(y)}{|E(X)|}, & \text{if } X \text{ is not bipartite,} \\ \lim_{n \rightarrow \infty} q^{(2n+\delta)}(x, y) &= \frac{2 \deg(y)}{|E(X)|}, & \text{if } X \text{ is bipartite,} \end{aligned}$$

where  $\delta = 0$  (resp.  $\delta = 1$ ) according to whether  $x$  and  $y$  are at even (resp. odd) distance.

(c) *If  $X$  is infinite, connected, with minimum degree 2, then for all  $x, y \in X$ ,*

$$\lim_{n \rightarrow \infty} q^{(n)}(x, y) = 0.$$

In statements (a) and (b), note that  $|E(X)|$  is twice the number of non-oriented edges.

As usual, the *distance*  $d(x, y)$  between two vertices  $x, y \in X$  is the minimum length of a path connecting the two. The *ball* of radius  $R$  centred at  $x$  is the subgraph  $B(x, R) = \{y \in X : d(y, x) \leq R\}$  of  $X$ . Recall that a *cycle* of length  $n$  in  $X$  consists of a sequence  $e_n = e_0, \dots, e_{n-1}$  of distinct edges whose initial vertices are all distinct, such that  $e_{k-1} \rightarrow e_k$  for all  $k = 1, \dots, n$ .

**(1.3) Definition.** We say that *small cycles are dense in  $X$* , if there is  $R > 0$  such that every ball  $B(x, R)$  in  $X$  contains a cycle.

Every finite, connected graph with minimum degree 2 satisfies this condition.

The *automorphism group* of  $X$  consists of all bijections  $g : X \rightarrow X$  which satisfy  $e(gx, gy) = e(x, y)$  for all  $x, y \in X$ . A graph is called *transitive*, resp. *almost transitive* if the automorphism group acts with one orbit, resp. finitely many orbits on  $X$ . Obviously, an infinite, almost transitive graph with minimum degree 2 has dense small cycles unless it is a tree. (To be precise, we require of a tree that it does not have multiple edges.)

**(1.4) Lemma.** *If small cycles are dense in  $X$  then*

$$\rho(Q) = \limsup_{n \rightarrow \infty} q^{(n)}(x, y)^{1/n}$$

*is independent of  $x, y \in X$ , and  $0 < \rho(Q) \leq 1$ . (If  $X$  is finite then  $\rho(Q) = 1$ .)*

The following strengthens Theorem 1.2 (c) for almost transitive graphs.

**(1.5) Theorem.** *If  $X$  is infinite, connected, with minimum degree 2, and almost transitive, then for all  $x, y \in X$ ,*

$$\lim_{n \rightarrow \infty} q^{(n)}(x, y) / \rho(Q)^n = 0.$$

The *isoperimetric constant*  $\iota(X)$  of a connected, locally finite graph  $X$  is

$$\iota(X) = \inf \left\{ \frac{\text{Area}(F)}{\text{Vol}(F)} : F \subset X \text{ finite} \right\},$$

where  $\text{Vol}(F) = \sum_{x \in F} \deg(x)$  and  $\text{Area}(F)$  is the number of edges with one endpoint in  $F$  and the other in  $X \setminus F$ . The graph is called *amenable* if  $\iota(X) = 0$ . Non-amenable graphs are also called (infinite) *expanders*.

Consider the Hilbert space  $\ell^2(E)$  of all functions  $F : E \rightarrow \mathbb{R}$  with  $\langle F, F \rangle < \infty$ , with the ordinary inner product

$$\langle F, G \rangle = \sum_{e \in E} F(e)G(e).$$

Then  $Q_E$  acts on this space by  $Q_E F(e) = \sum_{f \in E} q_E(e, f)F(f)$ . We denote by  $\|Q_E\|$  the corresponding operator norm, and by  $\rho_2(Q_E) = \lim_n \|Q_E^n\|^{1/n}$  its spectral radius. Note that  $\rho(Q) \leq \rho_2(Q_E) \leq \|Q_E\|$  in general.

**(1.6) Proposition.** (a) *One has always  $\|Q_E\| = 1$ .*

(b) *If small cycles are dense in  $X$ , then  $\rho(Q) = \rho_2(Q_E)$ .*

**(1.7) Theorem.** *Suppose that  $X$  is connected, that small cycles are dense, and that there is  $M < \infty$  such that  $2 \leq \deg(x) \leq M$  for all  $x \in X$ .*

*Then  $X$  is amenable if and only if  $\rho(Q) = 1$ .*

With these results and their proofs we aim principally at extending and explaining previous material regarding *cogrowth* of graphs and groups and at shedding new light on cogrowth by studying it in terms of NBRW on the oriented edges. We also think that NBRW on the (oriented) edge set of an arbitrary graph is an interesting random process in its own right.

In §2, we first recall (ordinary) simple random walk on a graph and some of its basic properties in order to put our results on NBRW in the right perspective. We then consider cogrowth of graphs, which is best understood in terms of universal covering trees, and explain how Theorems 1.2, 1.5 and 1.7 apply. In §2 we also give various references.

§3 is dedicated to the proofs of the results stated here.

Some additional remarks and observations can be found in §4.

## 2. SIMPLE RANDOM WALK, AND COGROWTH OF GRAPHS

**A. Simple random walk** (SRW) is mostly considered on graphs without multiple edges, and loops are usually counted only once for the degree of a vertex. Here, multiple edges are admitted, and we count each loop twice. SRW is the Markov chain on the (vertex set of the) graph  $X$  with transition matrix  $P = (p(x, y))_{x, y \in X}$  given by

$$p(x, y) = \frac{e(x, y)}{\deg(x)}.$$

Thus, contrary to NBRW, the walker does not remember from where he did come at the previous step, and chooses at random anyone among the outgoing edges at the actual vertex. A possible interpretation for counting each loop twice is that topologically, the walker standing at a vertex  $x$  sees two “ends” of each loop at  $x$  among which he may choose. We write  $p^{(n)}(x, y)$  for the  $n$ -step transition probability from  $x$  to  $y$ .

The transition matrix  $P$  acts by  $Pg(x) = \sum_y p(x, y)g(y)$  on the Hilbert space  $\ell^2(X, \deg)$  of all functions  $g : X \rightarrow \mathbb{R}$  with  $\langle g, g \rangle < \infty$ , where the inner product is

$$\langle g, h \rangle = \sum_{x \in X} g(x)h(x) \deg(x).$$

We denote by  $\|P\|$  the norm of this operator.

Here is a list of well-known properties of SRW. (Recall once more that  $E = E(X)$  is the set of oriented edges as in §1, so that  $|E(X)|$  is twice the number of “ordinary” non-oriented edges.)

**(2.1) Proposition.** *Let  $X$  be a connected, locally finite graph.*

(a) *If  $X$  is finite and not bipartite, then for all  $x, y \in X$ ,*

$$\lim_{n \rightarrow \infty} p^{(n)}(x, y) = \frac{\deg(x)}{|E(X)|}.$$

*If  $X$  is finite and bipartite, then for all  $x, y \in X$ , with  $\delta \in \{0, 1\}$  such that  $d(x, y) \equiv \delta \pmod{2}$ ,*

$$\lim_{n \rightarrow \infty} p^{(2n+\delta)}(x, y) = 2 \frac{\deg(x)}{|E(X)|}.$$

(b) *If  $X$  is infinite, then for all  $x, y \in X$ ,*

$$\lim_{n \rightarrow \infty} p^{(n)}(x, y) = 0.$$

(c) *The spectral radius*

$$\rho(P) = \limsup_{n \rightarrow \infty} p^{(n)}(x, y)^{1/n}$$

*is independent of  $x, y \in X$ , and  $\|P\| = \rho(P)$ .*

(d) *If  $X$  is infinite and almost transitive then*

$$\lim_{n \rightarrow \infty} p^{(n)}(x, y) / \rho(P)^n = 0.$$

(e)  *$X$  is amenable if and only if  $\rho(P) = 1$ .*

Statements (a) and (b) follow from basic Markov chain theory, see e.g. CHUNG [2] or SENETA [14]: the Markov chain given by  $P$  is *irreducible* ( $\forall x, y \in X \exists n = n(x, y) \geq 0$  such that  $p^{(n)}(x, y) > 0$ ). Its *period*  $\mathfrak{d}(P) = \gcd\{n : p^{(n)}(x, x) > 0\}$  is  $= 2$  when  $X$  is bipartite, and  $= 1$ , otherwise. Finally,  $\mu(x) = \deg(x)$  defines an *invariant* measure. If  $X$  is finite then  $\mu(X) = |E(X)|$ , and  $\mu_0(x) = \mu(x)/|E(X)|$  is an invariant probability measure. Therefore, (a) follows from the basic convergence theorem, see [2], Thm. 1 in §I.6 or [14], Thm. 4.2. If  $X$  is infinite then  $\mu(X) = \infty$ , whence the random walk cannot be *positive recurrent*, and (b) must hold. We shall encounter these notions in more detail in §3.

For statement (c), see e.g. WOESS [17], §10. In particular, the fact that  $\rho(P) = \rho_2(P)$ , the  $\ell^2$ -spectral radius of  $P$ , follows from self-adjointness of  $P$  on  $\ell^2(X, \deg)$ .

Regarding statement (d), this is immediate when  $\sum_n p^{(n)}(x, y) / \rho(P)^n < \infty$ . If the series diverges then it follows from Theorem 7.8 in [17] (which is basically due to GUIVARC'H [7]) that  $\rho(P) = 1$ , and we can apply (b).

Statement (e) has a long history, going back to KESTEN's amenability criterion for finitely generated groups [10]. The version stated here is due to DODZIUK AND KENDALL [5] based on a previous paper by DODZIUK [4].

**B. Cogrowth** is a notion of asymptotic density of a graph. It is best understood in terms of the *universal cover* of the graph  $X$ . This is a (unique) *tree*  $T$  together with a surjective mapping  $\pi : T \rightarrow X$  which is a local homeomorphism, i.e., if  $\tilde{x}, \tilde{y}$  are neighbours in  $T$  then so are  $\pi(\tilde{x}), \pi(\tilde{y})$  in  $X$ , and  $\deg_T(\tilde{x}) = \deg_X(\pi(\tilde{x}))$  for every vertex  $\tilde{x} \in T$ .

The covering tree can be constructed as follows: a non-backtracking walk of length  $n \geq 0$  in  $X$  is a sequence  $e_1, \dots, e_n$  of edges such that  $e_{k-1} \rightarrow e_k$  for  $k = 2, \dots, n$ . Its initial and terminal vertices are  $e_1^-$  and  $e_n^+$ , respectively. If  $n = 0$ , we have an empty path, for which we have to specify its initial = terminal vertex. We now choose a root (reference vertex)  $o \in X$ , and define  $T$  as the set of all non-backtracking paths  $\tilde{x}$  starting at  $o$ , including the empty path. Two such paths are defined to be neighbours in  $T$  if one of them extends the other by a single edge. The mapping  $\pi$  assigns to each  $\tilde{x} \in T$  its terminal vertex  $x \in X$ .

Now let  $x, y \in X$ , and choose  $\tilde{x} \in T$  such that  $\pi(\tilde{x}) = x$ . Write  $T(y) = \{\tilde{y} \in T : \pi(\tilde{y}) = y\}$ , and consider the sphere  $S(\tilde{x}, n) = \{\tilde{v} \in T : d_T(\tilde{v}, \tilde{x}) = n\}$ , where  $d_T(\cdot, \cdot)$  is the

distance in  $T$ . Then (*ordinary*) *cogrowth* at  $x, y \in X$  is the sequence

$$(2.2) \quad \text{cog}_n(x, y) = \frac{|S(\tilde{x}, n) \cap T(y)|}{|S(\tilde{x}, n)|}, \quad n \geq 0.$$

The graph  $X$  being “small” corresponds to  $(\text{cog}_n(x, y))_n$  being “large”. Besides finiteness, also amenability is a “smallness” condition, whence it is natural to look for a link between cogrowth and amenability.

Cogrowth was initially introduced by GRIGORCHUK [6] and later COHEN [3] for finitely generated *groups*. If  $\Gamma$  is such a group, then we can represent it as a factor  $\mathbb{F}_s/N$ , where  $\mathbb{F}_s$  is the *free group* on  $s$  free generators  $\tilde{a}_1, \dots, \tilde{a}_s$ , and  $N$  is a normal subgroup of  $\mathbb{F}_s$ . Let  $\pi : \mathbb{F}_s \rightarrow \Gamma$  be the factor map. We write  $\tilde{a}_{-i} = \tilde{a}_i^{-1}$  and set  $\tilde{S} = \{\tilde{a}_i : i = \pm 1, \dots, \pm s\}$ . Then the Cayley graph of  $\mathbb{F}_s$  with respect to  $\tilde{S}$  is the  $2s$ -regular tree, which is the covering tree of the Cayley graph of  $\Gamma$  with respect to the generators  $a_i = \pi(\tilde{a}_i)$ . It is best to consider immediately the oriented edges of that Cayley graph: every  $x \in \Gamma$  is the initial point of an edge of type  $\tilde{a}_i$ , whose endpoint is  $xa_i$ ; the associated “inverse” edge goes from  $xa_i$  to  $x$  and has type  $\tilde{a}_{-i}$  ( $i = \pm 1, \dots, \pm s$ ). Every pair of this type corresponds to one unoriented edge. Note that generators with  $a_i = a_{-i} \neq id$  give rise to multiple edges, and when  $a_i = a_{-i} = id$ , we get loops. This also explains why loops should be counted twice for the degrees. Thus, the factor map  $\pi$  becomes the covering map from the tree onto the Cayley graph.

Note that for groups,  $\text{cog}_n(x, x)$  is the same for all  $x$ . Amenability of a finitely generated group  $\Gamma$  is equivalent with amenability of any of its (locally finite) Cayley graphs. The main result of [6] and [3], restated in our notation, was that

$$(2.3) \quad \Gamma \text{ is amenable} \iff \limsup_{n \rightarrow \infty} \text{cog}_n(x, x)^{1/n} = 1.$$

This has been generalized to regular graphs by NORTHSHIELD [11], who was also the first to explain cogrowth in terms of covering trees. One of the basic tools for studying cogrowth of regular graphs is a functional equation between the generating functions  $C(x, y|t) = \sum_n \text{cog}_n(x, y) t^n$  of the cogrowth sequence and  $G(x, y|z) = \sum_n p^{(n)}(x, y) z^n$  of the transition probabilities of SRW: if  $X$  is  $d$ -regular then with our notation and normalizations,

$$(2.4) \quad C(x, y|t) = \frac{1}{d} \delta_x(y) + \frac{(d-1)^2 - t^2}{d(d-1+t^2)} G(x, y|z(t)), \quad \text{where} \quad z(t) = \frac{dt}{d-1+t^2},$$

A first version of (2.4) is contained in the Ph.D. thesis of GRIGORCHUK. Various proofs of that formula have appeared: WOESS [16], SZWARC [15] (both for groups), NORTHSHIELD [11] (shortest), BARTHOLDI [1] (more general). In spite of [1], there is no satisfactory version of that formula for non-regular graphs. Nevertheless, NORTHSHIELD [12] proves a clever extension of (2.3) to *quasi-regular* graphs (non-regular graphs satisfying a certain uniform growth condition), and studies cogrowth of arbitrary graphs under certain restrictions [13].

More generally, we can consider a sequence  $\nu = (\nu_{\tilde{x}, n})_{\tilde{x} \in T, n \geq 0}$ , where each  $\nu_{\tilde{x}, n}$  is a probability measure concentrated on the sphere  $S(\tilde{x}, n)$  of radius  $n$  centred at  $\tilde{x}$  in the

covering tree  $T$  of  $X$ , with  $\pi(\tilde{x}) = x$ . Note that there is a natural bijection between  $S(\tilde{x}, n)$  and  $S(\tilde{x}', n)$ , when  $\pi(\tilde{x}) = \pi(\tilde{x}')$ . We require that in this case,  $\nu_{\tilde{x}', n}$  is the image of  $\nu_{\tilde{x}, n}$  under that bijection. Then we can define

$$(2.5) \quad \text{cog}_n^\nu(x, y) = \nu_{\tilde{x}, n}(T(y)), \quad x, y \in X, \pi(\tilde{x}) = x.$$

When each  $\nu_{\tilde{x}, n}$  is equidistribution on  $S(\tilde{x}, n)$ , this is ordinary cogrowth.

Another choice is to define

$$\nu_{\tilde{x}, n}(\tilde{y}) = \frac{1}{\deg(\tilde{x})} \frac{1}{\deg(\tilde{x}_1) - 1} \cdots \frac{1}{\deg(\tilde{x}_{n-1}) - 1},$$

where  $\tilde{x}, \tilde{x}_1, \dots, \tilde{x}_{n-1}, \tilde{y}$  are the consecutive vertices on the unique path in  $T$  from  $\tilde{x}$  to  $\tilde{y} \in S(\tilde{x}, n)$ . Cogrowth with respect to this choice of  $\nu$  is the same as NBRW:

$$(2.6) \quad \text{cog}_n^\nu(x, y) = q^{(n)}(x, y)$$

In the specific case of regular graphs, the two concepts coincide. Thus, besides ordinary cogrowth, non-backtracking random walk is another way to extend cogrowth from regular to arbitrary graphs.

### 3. PROOFS

In this section, we always use the basic assumption that  $X$  is a locally finite, connected graph with minimum degree 2. The following is straightforward.

**Proof of Lemma 1.1.** We have

$$\begin{aligned} q^{(n)}(x, y) &= \sum_{\substack{e_1 \in E: \\ e_1^- = x}} \frac{1}{\deg(x)} \sum_{\substack{e_2, \dots, e_n = f \in E: \\ e_{i-1}^- \rightarrow e_i, e_n^+ = y}} \frac{1}{\deg(e_1^+) - 1} \cdots \frac{1}{\deg(e_{n-1}^+) - 1} \\ &= \frac{1}{\deg(x)} \sum_{\substack{e_1, f \in E: \\ e_1^- = x, f^+ = y}} q^{(n-1)}(e_1, f) \\ &= \frac{1}{\deg(x)} \sum_{\substack{e_1, f \in E: \\ e_1^- = x, f^+ = y}} \frac{1}{\deg(x) - 1} \sum_{\substack{e \in E: \\ e \rightarrow e_1}} q^{(n-1)}(e, f) = \frac{1}{\deg(x)} \sum_{\substack{e, f \in E: \\ e^+ = x, f^+ = y}} q_E^{(n)}(e, f), \end{aligned}$$

since for any  $e_1 \in E$  with  $e_1^- = x$  there are  $\deg(x) - 1$  edges  $e$  with  $e \rightarrow e_1$ .  $\square$

It may be best to think of edge-NBRW as simple random walk on the *oriented line graph* (OLG) of  $X$ . This is the digraph whose vertex set is  $E = E(X)$ , and there is an oriented (2nd order) edge from  $e$  to  $f$  ( $e, f \in E$ ) if  $e \rightarrow f$ . Our Markov chain with transition matrix  $Q_E$  is not symmetric, nor reversible like SRW on an unoriented graph. However, the counting measure  $\lambda$ , given by  $\lambda(e) = 1$ , is an invariant measure for  $Q_E$ , that is,

$$(3.1) \quad \sum_{e \in E} \lambda(e) q_E(e, f) = \lambda(f) \quad \forall f \in E.$$

We now recall a few basic Markov chain notions. We write  $e \xrightarrow{*} f$  if there is  $n \geq 0$  such that  $q_E^{(n)}(e, f) > 0$  (i.e., there is an oriented path from  $e$  to  $f$  in the OLG, a transitive

relation), and  $e \overset{*}{\leftrightarrow} f$  if  $e \overset{*}{\rightarrow} f$  and  $f \overset{*}{\rightarrow} e$ . The equivalence classes with respect to the relation  $\overset{*}{\leftrightarrow}$  are called *irreducible classes*. An *essential class*  $V$  is an irreducible class with the property that  $e \in V$  and  $e \overset{*}{\rightarrow} f$  implies  $f \in V$ . Its elements are also called *essential*. The Markov chain and its transition matrix  $Q_E$  are called irreducible if the state space  $E$  forms a single irreducible class. (In graph theoretic terminology, this means that the OLG is *strongly connected*.)

**(3.2) Lemma.** *If  $X$  is finite then  $Q_E$  is irreducible, unless  $X$  is a cycle.*

*Proof.* Assume that  $X$  is not a cycle. Since  $X$  is connected, for any pair of edges  $e, f$ , at least one of  $e \overset{*}{\rightarrow} f$ ,  $e \overset{*}{\rightarrow} \check{f}$ ,  $\check{e} \overset{*}{\rightarrow} f$ , or  $\check{e} \overset{*}{\rightarrow} \check{f}$  must hold. Therefore it is sufficient to show that  $e \overset{*}{\rightarrow} \check{e}$  for every  $e \in E$ .

Let us first assume that  $e$  is not contained in any cycle of  $X$ . As  $\deg(x) \geq 2 \forall x$  we can find inductively a sequence  $e = e_0, e_1, e_2, \dots$  of edges such that  $e_{k-1} \rightarrow e_k$ . By finiteness of  $X$ , there must be a minimal index  $m$  such that  $e_m^+ = e_i^-$  for some  $i \in \{1, \dots, m-1\}$ . The edges  $e_i, \dots, e_m$  form a cycle  $C_1$ , so that

$$e = e_0 \overset{*}{\rightarrow} e_m \rightarrow \check{e}_{i-1} \overset{*}{\rightarrow} \check{e}_0 = \check{e}.$$

Now assume that  $e$  is contained in a cycle  $C_1$  formed by edges  $e = e_0, \dots, e_m$ . Since we are assuming that  $X$  is not a cycle, there is a vertex  $e_i^- =: x$  in  $C_1$  with  $\deg(x) \geq 3$ . Thus, there is an edge  $f$  with  $f^- = x$  such that  $f \notin \{\check{e}_{i-1}, e_i\}$  (for  $i = 0$  we intend  $e_{-1} = e_m$ ). If  $f$  does not lie on any cycle in  $X$ , we have already seen that  $f \overset{*}{\rightarrow} \check{f}$ , whence

$$e = e_0 \overset{*}{\rightarrow} e_{i-1} \rightarrow f \overset{*}{\rightarrow} \check{f} \rightarrow \check{e}_{i-1} \overset{*}{\rightarrow} \check{e}_0 = \check{e}.$$

On the other hand, assume that  $f$  is contained in a cycle  $C_2$  formed by edges  $f = f_0, \dots, f_\ell$ . Then there must be another edge  $f_k$  ( $k > 0$ ) incident with some vertex in  $C_1$ . Let  $j$  be the minimal index  $\in \{0, \dots, m\}$  with  $e_j^+ = f_k^+$  for some  $k \in \{1, \dots, \ell\}$ . Then

$$e \overset{*}{\rightarrow} e_{i-1} \rightarrow f = f_0 \overset{*}{\rightarrow} f_k \rightarrow \check{e}_j \overset{*}{\rightarrow} \check{e}. \quad \square$$

If  $X$  is a finite cycle, then the OLG consists of two disjoint, oriented cycles of the same length, each of which constitutes an essential class of  $Q_E$ , on which *NBRW* moves “forward” deterministically.

**(3.3) Lemma.** *If  $X$  is infinite then for any edge  $e \in E$  there are infinitely many edges  $f \in E$  with  $e \overset{*}{\rightarrow} f$ .*

*Proof.* Let  $e \in E$  and  $X'$  be the graph that results from  $X$  by removing  $e$  and  $\check{e}$ . If  $X'$  is connected then by infiniteness,  $e \overset{*}{\rightarrow} f$  for infinitely many  $f \in E$ . The same holds if  $e$  is directed towards an infinite component. Thus, let us assume that  $e$  is directed towards a finite component  $X'_1$  of  $X'$ . By infiniteness of  $X$ ,  $\check{e}$  is directed to the other, infinite component, so that  $\check{e} \overset{*}{\rightarrow} f$  for infinitely many  $f \in E$ . Applying the method of proof of Lemma 3.2 to  $X'_1$ , we have  $g \overset{*}{\rightarrow} \check{g}$  for some edge  $g$  with  $e \rightarrow g$  in  $X$  (remember that we assumed that  $\deg(e^+) \geq 2$ ). It follows that  $e \rightarrow g \overset{*}{\rightarrow} \check{g} \rightarrow \check{e}$  and hence  $e \overset{*}{\rightarrow} f$  for infinitely many  $f \in E$ .  $\square$



In general, if  $Q_E$  is irreducible, then we can define its *period* by

$$\mathfrak{d} = \mathfrak{d}(Q_E) = \gcd\{n : q_E^{(n)}(e, e) > 0\},$$

which is independent of  $e \in E$ .

**(3.4) Lemma.** *Let  $X$  be a finite, connected graph with  $\deg(x) \geq 3$  for all  $x \in X$ . Then the period of the associated edge-NBRW is either 2 or 1, depending on whether  $X$  is bipartite or not (respectively).*

*Proof.* First we shall show that  $\mathfrak{d}(Q_E) \in \{1, 2\}$ . Let  $e, f, g$  be three distinct edges with  $e^- = f^- = g^- =: x$ . By assumption,  $\deg(e^+) \geq 3$  and there are two distinct edges  $e_1, e_2 \neq \check{e}$  with  $e^+ = e_1^- = e_2^-$ . By Lemma 3.2 we have  $e_1 \xrightarrow{*} \check{f}$  and  $e_2 \xrightarrow{*} \check{g}$ . Then there are non-backtracking closed paths such that  $f \xrightarrow{*} \check{e}_1 \rightarrow \check{e} \rightarrow f$  in (say)  $m$  steps and  $g \xrightarrow{*} \check{e}_2 \rightarrow \check{e} \rightarrow g$  in (say)  $n$  steps. But we also have  $f \xrightarrow{*} f$  in  $n + m - 2$  steps via

$$f \xrightarrow{*} \check{e}_1 \rightarrow e_2 \xrightarrow{*} \check{g} \rightarrow f.$$

Therefore,  $\mathfrak{d}(Q_E)$  must be a factor of  $n$ ,  $m$  and  $n + m - 2$ , whence  $\mathfrak{d}(Q_E) \in \{1, 2\}$ .

It is now clear that we must have  $\mathfrak{d}(Q_E) = 2$ , if  $X$  is bipartite. Otherwise,  $X$  contains an odd cycle, so that  $q_E^{(k)}(e, e) > 0$  for some odd  $k$ . Thus, we cannot have  $\mathfrak{d}(Q_E) = 2$ , that is,  $\mathfrak{d}(Q_E) = 1$ .  $\square$

**Proof of Theorem 1.2.** (a+b) If  $X$  is finite, but not a cycle, then we can use Lemma 3.2. Let  $e, f \in E$  and  $r \geq 0$  such that  $q_E^{(r)}(e, f) > 0$ . Then  $q_E^{(n)}(e, f) > 0$  if and only if  $n \equiv r \pmod{\mathfrak{d}}$  and  $n$  is sufficiently large (see [14], Thm. 1.3). The fundamental convergence theorem (see [2], Thm. 1 in §I.6 or [14], Thm. 4.2) implies that

$$(3.5) \quad \lim_{n \rightarrow \infty} q_E^{(n\mathfrak{d}+r)}(e, f) = \mathfrak{d} \lambda_0(f) = \frac{\mathfrak{d}}{|E(X)|},$$

where  $\lambda_0$  is the unique invariant probability measure, that is,  $\lambda_0(f) = \frac{1}{|E|}$ . In view of Lemma 3.4, this together with Lemma 1.1 yields statement (b), when  $\deg(x) \geq 3$  for all  $x \in X$ .

Otherwise,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left( q_E^{(1)}(e, f) + \cdots + q_E^{(n)}(e, f) \right) = \frac{1}{|E(X)|},$$

and combining this with Lemma 1.1, we obtain the limit proposed in statement (a) of Theorem 1.2.

In the case where  $X$  is a cycle the  $q^{(n)}(x, y)$  can be calculated explicitly, whence the claim of the Theorem follows. This is left as a simple exercise to the reader.

(c) We distinguish two cases. First, if the edge-NBRW starting at  $e \in E$  is *transient*, that is, the probability of returning to  $e$  is  $< 1$ , then  $\sum_n q^{(n)}(e, f) < \infty$  for every  $f \in E$ , see [2], Thm. 4 in §I.6. Therefore,  $q^{(n)}(e, f) \rightarrow 0$ .

If the random walk starting at  $e$  is *recurrent*, i.e., it returns to  $e$  with probability 1, then  $e$  must be an essential state, see [2], Thm. 4 in §I.4 or [14], Lemma 5.2. Now by Lemma 3.3, there are infinitely many  $f \in E$  such that  $e \xrightarrow{*} f$ . Therefore, the – essential – irreducible class  $V$  of  $e$  is infinite. Since the random walk starting at  $e$  does not leave  $V$ , we

can consider the restriction of  $Q_E$  to  $V$ . It defines an irreducible, recurrent Markov chain with invariant measure  $\lambda$ , the counting measure. Recurrence yields that this is the unique invariant measure up to normalization. It has total mass  $\lambda(V) = \infty$ , the chain is *null recurrent*, see [2], §I.6 or [14], §§5.2–5.3. Therefore the convergence theorem for recurrent Markov chains yields that  $q_E^{(n)}(e, f) \rightarrow 0$  for all  $f \in V$ . If  $f \notin V$  then  $q_E^{(n)}(e, f) = 0$  for all  $n$ . Since  $X$  is by assumption locally finite, formula Lemma 1.1 yields the result stated in (c).  $\square$

**Uniformly irreducible random walks and amenability.** We now make a small detour regarding more general random walks on graphs, recalling and improving upon the material in [17], §10.B.

Let  $X$  be a locally finite, connected graph with graph metric  $d(\cdot, \cdot)$ , and consider the transition matrix  $P = (p(x, y))_{x, y \in X}$  of an arbitrary random walk (Markov chain) on the set  $X$ . Then  $P$  is called *uniformly irreducible* if there are constants  $K, \varepsilon_0 > 0$  such that for any pair of neighbours  $x, y$  there is some  $k \leq K$  such that  $p^{(k)}(x, y) \geq \varepsilon_0$ . Furthermore,  $P$  is said to have *bounded range*, if there is  $R > 0$  such that  $p(x, y) > 0$  only if  $d(x, y) \leq R$ . These two are conditions of adaptedness of  $P$  to the graph structure.

If  $P$  has an *invariant measure*  $\nu$ , then it acts on the Hilbert space  $\ell^2(X, \nu)$  of all  $F : X \rightarrow \mathbb{R}$  with  $\langle F, F \rangle < \infty$ , where  $\langle F, G \rangle = \sum_x F(x)G(x)\nu(x)$ . The operator norm satisfies  $\|P\| \leq 1$ , and its  $\ell^2$ -spectral radius is  $\rho_2(P) = \lim_n \|P^n\|^{1/n}$ . Note that for  $\rho(P) = \limsup_n p^{(n)}(x, y)^{1/n}$  (independent of  $x, y$  by irreducibility) one has  $\rho(P) \leq \rho_2(P)$ , and equality does not hold in general. The adjoint (more precisely,  $\nu$ -adjoint)  $P^*$  of  $P$  on  $\ell^2(X, \nu)$  has the stochastic kernel  $p^*(x, y) = \nu(y)p(y, x)/\nu(x)$ .

**(3.6) Theorem.** *Suppose that  $X$  is connected, with bounded vertex degrees, and that  $P$  is uniformly irreducible with bounded range and has an invariant measure  $\nu$  satisfying  $C^{-1} \leq \nu(\cdot) \leq C$  for some  $C \geq 1$ .*

*Then  $\rho_2(P) = 1$  if and only if the graph  $X$  is amenable.*

*Proof (outline).* Theorem 10.6 in [17] states that under the given assumptions,  $\rho(P) = 1$  implies amenability of  $X$ . After the proof of that theorem, it is explained that the condition  $\rho(P) = 1$  may be replaced with  $\rho_2(P) = 1$ .

Conversely, Theorem 10.8 in [17] states that amenability of  $X$  implies  $\|P\| = 1$ . Now, let  $I$  be the identity operator (or matrix), and fix  $n \geq 1$ . Set  $\bar{P} = \frac{1}{2}(I + P)$ . Then  $\bar{P}^n$  is uniformly irreducible, has bounded range and invariant measure  $\nu$ . If  $X$  is amenable, then we get that  $\|\bar{P}^n\| = 1$ . This is true for every  $n$ . Consequently,  $\rho_2(\bar{P}) = 1$ . By basic spectral theory, also  $\rho_2(P) = 1$ .  $\square$

More generally, the bounded range assumption can be replaced with tightness of the step length distributions of  $P$  and  $P^*$  as in [17], Thm. 10.8.

We want to apply Theorem 3.6 not to random walks on our “original” graph  $X$ , but to edge-NBRW on the OLG. However, the latter is not a graph (with unoriented edges), but a digraph. Therefore, we symmetrize it by “removing the arrows” from its edges. (Recall

that the latter are “second order” edges, connecting edges of the original graph  $X$ ). The resulting SOLG (symmetrized oriented line graph) still has as its vertex set the set  $E$  of *oriented* edges of the original graph  $X$ , but neighbourhood in the SOLG is given by  $e \sim f$ , if  $e \rightarrow f$  or  $f \rightarrow e$ . We observe that in the SOLG,  $q_E(e, f) > 0$  implies  $e \sim f$ , but *not conversely*.

**(3.7) Lemma.** *If  $2 \leq \deg(x) \leq M$  for all  $x \in X$ , and small cycles are dense in  $X$ , then there is  $L > 0$  such that for each  $e \in E$ , we have  $e \xrightarrow{*} \check{e}$  in at most  $L$  steps of edge-NBRW. In particular,  $Q_E$  is uniformly irreducible on the symmetrized OLG.*

*Proof.* We may suppose that  $X$  is infinite. Observe that the first statement of the lemma implies uniform irreducibility. Indeed, let  $f$  be a neighbour of  $e$  in the SOLG. Then either  $e^+ = f^-$ , in which case  $q_E(e, f) \geq 1/(M-1)$ , or  $f^+ = e^-$ , in which case  $e \xrightarrow{*} \check{e} \rightarrow \check{f} \xrightarrow{*} f$  in  $k \leq 2L+1$  steps with probability  $\geq 1/(M-1)^{2L+1}$ .

Now let  $R > 0$  be such that  $B(x, R)$  contains a cycle for every  $x \in X$ . Since the vertex degree in  $X$  is bounded by  $M$ , the number of (oriented) edges in each  $B(x, R)$  cannot exceed a certain constant  $K = K(M, R)$ . Given  $e \in E$ , by Lemma 3.3 there are infinitely many edges  $f$  with  $e \xrightarrow{*} f$ . Thus, there is a non-backtracking path in  $X$  whose first edge is  $e$  and whose last edge  $f$  is the first one not lying in  $B(e^-, R)$ . We may assume that this path has no repeated edges, so that its length (number of edges) is at most  $K+1$ . Thus,  $e \xrightarrow{*} f$  in at most  $K$  steps of the edge-NBRW. By assumption  $B(f^+, R)$  contains a cycle  $C_1$  formed by edges  $e_1, \dots, e_m$  ( $m \leq K$ ). Since  $d(e^-, f^+) > R$  neither  $e$  nor  $\check{e}$  are edges inside the ball  $B(f^+, R)$  in  $X$ , and consequently neither of the two is among the edges  $e_1, \dots, e_m$  of  $C_1$ . Now, either  $f \xrightarrow{*} e_i$  (case 1) or  $\check{f} \xrightarrow{*} e_i$  (case 2) for some  $i \in \{1, \dots, m\}$  in at most  $R$  steps. If  $f$  or  $\check{f} = e_i$  for  $i \in \{1, \dots, m\}$ , then

$$\begin{aligned} & e \xrightarrow{*} f = e_i \rightarrow e_{i+1} \rightarrow \dots \rightarrow e_m \rightarrow e_1 \rightarrow \dots \rightarrow e_{i-1} \xrightarrow{*} \check{e} \\ \text{or} \quad & e \xrightarrow{*} f = \check{e}_i \rightarrow \check{e}_{i+1} \rightarrow \dots \rightarrow \check{e}_m \rightarrow \check{e}_1 \rightarrow \dots \rightarrow \check{e}_{i-1} \xrightarrow{*} \check{e}, \end{aligned}$$

respectively, in  $\leq K + K + K = 3K$  steps. Now let us assume that  $f, \check{f} \neq e_i$  for  $i \in \{1, \dots, m\}$ . Then we have in case 1

$$e \xrightarrow{*} f \xrightarrow{*} e_i \rightarrow e_{i+1} \rightarrow \dots \rightarrow e_m \rightarrow e_1 \rightarrow \dots \rightarrow e_{i-1} \xrightarrow{*} \check{f} \xrightarrow{*} \check{e}$$

in  $\leq K + R + K + R + K = 2R + 3K$  steps. In case 2 we have to turn off on the way to  $f$  to arrive at the cycle  $C_1$ . More exactly, let  $e = f_0, \dots, f_n = f$  be a walk from  $e$  to  $f$  in  $n \leq K$  steps. Now consider a walk from  $\check{f} = \check{f}_n$  to  $e_i$  in  $\leq R$  steps. It contains some (at least one) of the edges  $\check{f}_n, \check{f}_{n-1}, \dots, \check{f}_1$ . Let  $\ell$  be the minimal index such that  $\check{f}_\ell$  is not contained in the walk. Then we have

$$e \xrightarrow{*} f_\ell \xrightarrow{*} e_i \rightarrow e_{i+1} \rightarrow \dots \rightarrow e_m \rightarrow e_1 \rightarrow \dots \rightarrow e_{i-1} \xrightarrow{*} \check{f}_\ell \xrightarrow{*} \check{e}$$

again in  $\leq K + R + K + R + K = 2R + 3K$  steps. Thus setting  $L = 2R + 3K$  we have  $e \xrightarrow{*} \check{e}$  in  $\leq L$  steps.  $\square$

**Proof of Lemma 1.4.** If  $X$  is not a cycle then by Lemmas 3.2 and 3.7,  $Q_E$  is irreducible and a standard argument (see e.g. [14], Thm. 6.1, or [17], §1.B) yields that

$$(3.8) \quad \rho(Q_E) = \limsup_n q_E^{(n)}(e, f)^{1/n}$$

is independent of  $e, f \in E$ . Since  $1/\rho(Q_E)$  is the radius of convergence of each of the power series with non-negative coefficients  $\sum_n q_E^{(n)}(e, f) z^n$  ( $z \in \mathbb{C}$ ), where  $e, f \in E$ , Lemma 1.1 implies

$$\limsup_{n \rightarrow \infty} q^{(n)}(x, y)^{1/n} = \rho(Q_E)$$

for all  $x, y \in X$ , and Lemma 1.4 follows.

If  $X$  is a cycle of length  $k$ , then (without  $n$ -th roots)

$$\limsup_n q^{(n)}(x, y) = \begin{cases} 1, & \text{if } d(x, y) = 0 \text{ or } = k/2, \\ 1/2, & \text{otherwise,} \end{cases}$$

whence  $\rho(Q) = 1$  is independent of  $x, y$  as well.  $\square$

The fact that  $\rho(Q) = \rho(Q_E)$ , as stated in (3.8), is immediate from Lemma 1.1 and will be tacitly used several times.

**Proof of Theorem 1.5.** If  $X$  is a tree then for each pair  $e, f \in E$  there is at most one  $n$  such that  $q_E^{(n)}(e, f) > 0$ .

Otherwise,  $X$  has a cycle, and since it is almost transitive, small cycles are dense in  $X$ . By Lemma 1.4,  $Q_E$  is irreducible, and the OLG of  $X$  is connected. Therefore the series  $\sum_n q_E^{(n)}(e, f)/\rho(Q)^n$  either converge for all  $e, f$  or diverge for all  $e, f \in E$ , see e.g. [17], §1.B.

In the convergent case,  $q_E^{(n)}(e, f)/\rho(Q)^n \rightarrow 0$ .

In the divergent case, edge-NBRW is  $\rho$ -recurrent. The automorphism group  $\Gamma$  of  $X$  also acts with finitely many orbits on the OLG. Therefore we can apply an adaptation of a result of GUIVARC'H [7], see [17], Thm. 7.8 and its proof: it yields that there is a positive function  $H$  on  $E$  such that  $Q_E H = \rho(Q) \cdot H$ , and

$$q_H(e, f) = \frac{q_E(e, f)H(f)}{\rho(Q)H(e)}$$

defines a new random walk which is  $\Gamma$ -invariant and recurrent. By Theorem 3.26 and Lemma 3.25 in [17],  $Q_H$  has an invariant measure which is constant on each  $\Gamma$ -orbit, and consequently has infinite total mass. Therefore,  $Q_H$  is null recurrent, and  $q_H^{(n)}(e, f) \rightarrow 0$  for all  $e, f$ . Since

$$q_H^{(n)}(e, f) = \frac{q_E^{(n)}(e, f)H(f)}{\rho(Q)^n H(e)},$$

we find that  $q_E^{(n)}(e, f)/\rho(Q)^n \rightarrow 0$ .  $\square$

A *rough isometry* between two metric spaces  $(X, d), (X', d')$  is a mapping  $\varphi : X \rightarrow X'$  with the following properties.

$$(3.9) \quad \begin{aligned} A^{-1}d(x, y) - A^{-1}B &\leq d'(\varphi x, \varphi y) \leq Ad(x, y) + B \quad \forall x, y \in X, \text{ and} \\ d'(x', \varphi X) &\leq B \quad \forall x' \in X', \end{aligned}$$

where  $A \geq 1$  and  $B \geq 0$ . In this case we say that the two spaces are *roughly isometric*.

**(3.10) Proposition.** *If  $X$  is a connected graph with  $2 \leq \deg(x) \leq M$  that is not a cycle and has dense small cycles, then it is roughly isometric with its symmetrized oriented line graph.*

*Proof.* Two finite connected graphs are always roughly isometric. Let us assume that  $X$  is infinite, with edge set  $E$ . Throughout this proof, we write  $d_X(\cdot, \cdot)$  for the graph distance in  $X$ , and  $d_E(\cdot, \cdot)$  for the graph distance in the SOLG of  $X$ . Define the mapping  $\varphi : E \rightarrow X$  by  $\varphi e = e^-$ . Evidently,  $\varphi$  is surjective and hence

$$(3.11) \quad d_X(x, \varphi(E)) = 0 \quad \text{for all } x \in X.$$

Now given two vertices  $x, y$  in  $X$  with  $d_X(x, y) = d$ , it is clear that two arbitrary edges  $e, f$  starting in  $x$  and  $y$ , respectively, have distance at least  $d$  in the SOLG of  $X$ . It follows that

$$(3.12) \quad d_X(\varphi e, \varphi f) \leq d_E(e, f).$$

On the other hand, we obtain also an upper bound for  $d_E(e, f)$ . Clearly, if  $e, f$  are oriented the ‘‘right way’’ we have  $e \xrightarrow{*} f$  in  $d_X(e^-, f^-)$  steps. If one of them is oriented the other way, by Lemma 3.7 it takes at most  $L$  steps to turn around, i.e. to reach  $\check{e}$  from  $e$ . Thus we have  $e \xrightarrow{*} f$  in at most  $2L + d_X(e^-, f^-)$  steps, so that

$$(3.13) \quad d_E(e, f) - 2L \leq d_X(\varphi e, \varphi f).$$

Now, setting  $A = 1$  and  $B = 2L$  and combining (3.11)–(3.13) yields (3.9).  $\square$

**Proof of Proposition 1.6.** (a) We have  $\|Q_E\| = \|Q_E^*Q_E\|^{1/2}$ , where the adjoint operator  $Q_E^*$  has kernel  $q_E^*(e, f) = q_E(f, e)$ . Let  $F : E \rightarrow \mathbb{R}$ , and let  $e \in E$ . Then

$$Q_E^*Q_E F(e) = \sum_{f \in E} \sum_{g \in E} q_E(g, e)q_E(g, f) F(f).$$

Thus,  $Q_E^*Q_E$  is a symmetric, stochastic operator that takes a weighted average of all values of  $F$  on each of the finite sets  $\{f \in E : f^- = e^-\}$ , where  $e \in E$ . Consequently, it has norm 1.

(b) Instead of  $Q_E$  we shall use the new transition operator  $\bar{Q}_E = \frac{1}{2}(I_E + Q_E)$ , where  $I_E$  is the identity operator. Of course, its invariant measure is again the counting measure on  $E$ , and  $\bar{Q}_E^* = \frac{1}{2}(I_E + Q_E^*)$ . If we fix  $n$ , then  $\bar{Q}_E^{*n}\bar{Q}_E^n$  is again doubly stochastic, has finite range, and all its matrix elements are bounded below by those of  $c_n Q_E$ , where  $c_n = n/4^n$ . Since  $Q_E$  is (uniformly) irreducible by Lemma 3.7, the same holds for  $\bar{Q}_E^{*n}\bar{Q}_E^n$ .

We shall now use the obvious, but crucial relation

$$(3.14) \quad q_E^{(n)}(e, f) = q_E^{(n)}(\check{f}, \check{e}),$$

which also holds for  $\bar{Q}_E^n$  in the place of  $Q_E^n$ . Lemma 3.7 implies that for every  $e \in E$ ,

$$\bar{q}_E^{(L)}(e, \check{e}) \geq 1/C, \quad \text{where } C = (2M)^L.$$

( $M$  is the upper bound on the vertex degrees.) Therefore, using (3.14),

$$\bar{q}_E^{*(n)}(e, f) = \bar{q}_E^{(n)}(f, e) = \bar{q}_E^{(n)}(\check{e}, \check{f}) \leq C^2 \bar{q}_E^{(L)}(e, \check{e}) \bar{q}_E^{(n)}(\check{e}, \check{f}) \bar{q}_E^{(L)}(\check{f}, f) \leq C^2 \bar{q}_E^{(n+2L)}(e, f).$$

In particular, we obtain that  $\bar{Q}_E^{*n} \bar{Q}_E^n \leq C^2 \bar{Q}_E^{2n+2L}$  matrix-elementwise.

Now, since  $\bar{Q}_E^{*n} \bar{Q}_E^n$  is symmetric (self-adjoint) and irreducible, Lemma 10.1 in [17] implies that its norm satisfies  $\|\bar{Q}_E^{*n} \bar{Q}_E^n\| = \rho(\bar{Q}_E^{*n} \bar{Q}_E^n)$ , the latter number being defined in the same way as in (3.8), but for the powers of  $\bar{Q}_E^{*n} \bar{Q}_E^n$ . Thus, if we take  $e \in E$ , then

$$\begin{aligned} \rho(\bar{Q}_E^{*n} \bar{Q}_E^n) &= \lim_{m \rightarrow \infty} \langle (\bar{Q}_E^{*n} \bar{Q}_E^n)^m \delta_e, \delta_e \rangle^{1/m} \leq \lim_{m \rightarrow \infty} C^2 \langle \bar{Q}_E^{(2n+2L)m} \delta_e, \delta_e \rangle^{1/m} \\ &= \lim_{m \rightarrow \infty} C^2 \bar{q}_E^{((2n+2L)m)}(e, e)^{1/m} \leq C^2 \rho(\bar{Q}_E)^{2n+2L}, \end{aligned}$$

since  $\bar{q}_E^{(k)}(e, e) \leq \rho(\bar{Q}_E)^k$  for all  $k \geq 0$  and  $e \in E$ , a well known fact, see e.g. [14], §6.1 or [17], Lemma 1.9. We infer that

$$\rho_2(\bar{Q}_E) = \lim_{n \rightarrow \infty} \|\bar{Q}_E^{*n} \bar{Q}_E^n\|^{1/2n} \leq \lim_{n \rightarrow \infty} (C^2 \rho(\bar{Q}_E)^{2n+2L})^{1/2n} = \rho(\bar{Q}_E).$$

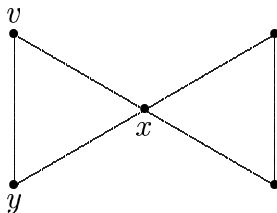
Since  $\rho(\bar{Q}_E) = \frac{1}{2}(1 + \rho(Q_E))$  and  $\rho_2(\bar{Q}_E) = \frac{1}{2}(1 + \rho_2(Q_E))$ , we conclude that  $\rho_2(Q_E) \leq \rho(Q_E)$ . The reversed inequality is obvious.  $\square$

**Proof of Theorem 1.7.** It is by now a well established fact that for connected graphs with bounded vertex degrees, amenability is rough-isometry-invariant. See e.g. [17], Thm. 4.7 (the isoperimetric inequality  $IS_\infty$  referred to there is the condition  $\iota(X) > 0$ , i.e., nonamenability), or also the book by DE LA HARPE [8]. Thus, in view of Proposition 3.10, under the assumptions of Theorem 1.7 the graph  $X$  is amenable if and only if its SOLG is amenable. By (3.1), edge-NBRW has the counting measure  $\lambda$  on  $E$  as an invariant measure, and by Lemma 3.7, it is uniformly irreducible. Therefore, we can apply Theorem 3.6 to the SOLG, and Proposition 1.6(b) allows us to replace the  $\ell^2$ -spectral radius with  $\rho(Q)$ .  $\square$

#### 4. FINAL REMARKS AND OBSERVATIONS

**A.** Regarding Theorem 1.2 (a+b), the condition  $\deg(x) \geq 3$  in Lemma 3.4 is necessary for the stronger convergence result of (b), as the following example shows. Thus, if there are vertices of degree  $\leq 2$  it is in general not true that for vertex-NBRW, one has convergence of  $q^{(2n+\delta)}(x, y)$  ( $\delta \in \{0, 1\}$ ) or  $q^{(n)}(x, y)$  according to whether  $X$  is bipartite or not (respectively).

**(4.1) Example.**



Clearly, edge-NBRW has period  $\mathfrak{d} = 3$ . Write  $e$  for the edge from  $y$  to  $x$  and  $f$  for the edge from  $v$  to  $y$ . We have

$$q^{(3n)}(x, x) = 1 \quad \text{and} \quad q^{(3n+1)}(x, x) = q^{(3n+2)}(x, x) = 0 \quad \forall n.$$

For the edges terminating at  $y$ , we have  $q_E^{(n)}(\check{e}, f) > 0$  only if  $n \equiv 1 \pmod 3$  and  $q_E^{(n)}(f, \check{e}) > 0$  only if  $n \equiv 2 \pmod 3$ , while  $q_E^{(n)}(\check{e}, \check{e})$  and  $q_E^{(n)}(f, f)$  are  $> 0$  only if  $n \equiv 0 \pmod 3$ . Therefore, using Lemma 1.1 and (3.5),

$$\begin{aligned} q^{(3n)}(y, y) &= \frac{1}{2} (q_E^{(3n)}(\check{e}, \check{e}) + q_E^{(3n)}(f, f)) \rightarrow \frac{1}{4}, \\ q^{(3n+1)}(y, y) &= \frac{1}{2} q_E^{(3n+1)}(\check{e}, f) \rightarrow \frac{1}{8}, \quad \text{and} \\ q^{(3n+2)}(y, y) &= \frac{1}{2} q_E^{(3n+2)}(f, \check{e}) \rightarrow \frac{1}{8}, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

**B.** For regular, almost transitive graphs, Lemma 3.9 of BARTHOLDI [1] states what is Proposition 2.1 (a)+(b)+(d) and Theorems 1.2+1.5 here. (We remark that in Lemma 3.9 of [1], the identity “ $\limsup_n \frac{g_n}{\beta^n} = \limsup_n \frac{f_n}{\alpha^n} = \dots$ ” should read “ $\limsup_n \frac{g_n}{\beta^n} = \frac{d}{d-1} \limsup_n \frac{f_n}{\alpha^n} = \dots$ ”.) In [1], a proof for SRW is suggested where one starts with the finite case, while for an infinite graph, one takes the sequence of balls  $B(o, r)$  around a “root” vertex, applies the “finite” result to each ball, and lets the radius tend to infinity, thereby exchanging two limits. [1] then suggests to use the same argument for cogrowth. This argument has also found its way into a recent paper of KAPOVICH ET AL. [9], who state an extension to arbitrary regular graphs. However, the argument is problematic because a priori it is by no means clear that the two limits (for  $n, r \rightarrow \infty$ ) may be exchanged.

As a matter of fact, this was the starting point for the present note, since several colleagues asked us how the mentioned argument can be made rigorous. When applied to regular graphs, our method provides a simple and rigorous proof of those statements for infinite graphs.

**C.** Theorems 1.2 and 1.5 extend the corresponding results for Cayley graphs of [16] to arbitrary graphs. At the same time, the functional equation (2.4) is no more needed. The extension of the amenability criterion (Theorem 1.7) required more work, since the functional equation (2.4) can be used only in the regular case. Also, in the regular case, that criterion does not require denseness of small circles. However, our result *is* a full generalization of that amenability criterion for (Cayley graphs of) finitely generated groups. Indeed, according to our definition of the Cayley graph, small circles will always

be dense in the latter unless the group is freely generated by the generating set that defines the Cayley graph. (Remember that when one of the generators satisfies  $a_i = a_i^{-1} \neq id$ , it leads to double edges. But double edges give rise to circles of length 2 according to our definition !)

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#### REFERENCES

- [1] Bartholdi, L.: *Counting paths in graphs*, Enseign. Math. (2) **45** (1999), no. 1-2, 83–131.
- [2] Chung, K. L.: *Markov Chains with Stationary Transition Probabilities*, Springer, Berlin, 1960.
- [3] Cohen, J. M.: *Cogrowth and amenability of discrete groups*, J. Funct. Anal. **48** (1982), 301–309.
- [4] Dodziuk, J.: *Difference equations, isoperimetric inequality, and transience of certain random walks*, Trans. Amer. Math. Soc. **284** (1984), 787–794.
- [5] Dodziuk, J., and Kendall, W. S.: *Combinatorial Laplacians and isoperimetric inequality*, in *From Local Times to Global Geometry, Control and Physics* (ed. K. D. Elworthy), Pitman Res. Notes Math. Ser. **150**, 1986, pp.68–74.
- [6] Grigorchuk, R. I.: *Symmetric random walks on discrete groups*, in *Multicomponent Random Systems* (eds. R. L. Dobrushin and Ya. G. Sinai), Nauka, Moscow 1978; English transl. in *Advances in Probability and Related Topics 6* (eds. D. Griffeath and P. Ney), M. Dekker, New York 1980, pp. 132–152.
- [7] Guivarc’h, Y.: *Sur la loi des grands nombres et le rayon spectral d’une marche aléatoire*, Astérisque **74** (1980), 47–98.
- [8] de la Harpe, P.: *Topics in Geometric Group Theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000.
- [9] Kapovich, I., Myasnikov, A., Schupp, P., and Shpilrain, V.: *Generic-case complexity, decision problems in group theory and random walks*, preprint, arXiv:math.GR/0203239 (2002).
- [10] Kesten, H.: *Full Banach mean values on countable groups*, Math. Scand. **7** (1959), 146–156.
- [11] Northshield, S.: *Cogrowth of regular graphs*, Proc. Amer. Math. Soc. **116** (1992), 203–205.
- [12] Northshield, S.: *Quasi-regular graphs, cogrowth, and amenability*, in *Dynamical systems and differential equations* (Wilmington, NC, 2002). Discrete Contin. Dyn. Syst. 2003, suppl., pp. 678–687.
- [13] Northshield, S.: *Cogrowth of arbitrary graphs*, in *Random walks and geometry* (ed. V. A. Kaimanovich with K. Schmidt and W. Woess), de Gruyter, Berlin 2004, pp. 501–513.
- [14] Seneta, E.: *Non-negative Matrices and Markov Chains*, Springer, Berlin, 1973.
- [15] Szwarz, R.: *A short proof of the Grigorchuk-Cohen cogrowth theorem*, Proc. Amer. Math. Soc. **106** (1989), 663–665.
- [16] Woess, W.: *Cogrowth of groups and simple random walks*, Arch. Math. (Basel) **41** (1983), 363–370.
- [17] Woess, W.: *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics **138**, Cambridge University Press, Cambridge, 2000.

INSTITUT FÜR MATHEMATIK C, TECHNISCHE UNIVERSITÄT GRAZ  
STEYRERGASSE 30, A-8010 GRAZ, AUSTRIA

*Current address of R. Ortner:*

Department Mathematik und Informationstechnologie, Montanuniversität Leoben  
Franz-Josef-Strasse 18, A-8700 Leoben, Austria

*E-mail addresses:* woess@TUGraz.at, ronald.ortner@unileoben.ac.at