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A comparative study of three boundary element approaches to calculate the transient response of viscoelastic solids with unbounded domains

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Abstract

As an alternative to domain discretization methods, the boundary element method (BEM) provides a powerful tool for the calculation of dynamic structural response in frequency and time domain. Field equations of motion and boundary conditions are cast into boundary integral equations (BIE), which are discretized only on the boundary. Fundamental solutions are used as weighting functions in the BIE which fulfil the Sommerfeld radiation condition, i.e., the energy radiation into a surrounding medium is modelled correctly. Therefore, infinite and semi-infinite domains can be effectively treated by the method. The soil represents such a semi-infinite domain in soil-structure-interaction problems. The response to vibratory loads superimposed to static pre-loads can often be calculated by linear viscoelastic constitutive equations. Conventional viscoelastic constitutive equations can be generalized by taking fractional order time derivatives into account. In the present paper two time domain BEM approaches including generalized viscoelastic behaviour are compared with the Laplace domain BEM approach and subsequent numerical inverse transformation. One of the presented time domain approaches uses an analytical integration of the elastodynamic BIE in a time step. Viscoelastic constitutive properties are introduced after Laplace transformation by means of an elastic–viscoelastic correspondence principle. The transient response is obtained by inverse transformation in each time step. The other time domain approach is based on the so-called 'convolution quadrature method'. In this formulation, the convolution integral in the BIE is numerically approximated by a quadrature formula whose weights are determined by the same Laplace transformed fundamental solutions used in the first method and a linear multistep method. A numerical study of wave propagation problems in 3-d viscoelastic continuum is performed for comparing the three BEM formulations.

1. Introduction

The boundary element method (BEM) has become a widely used numerical tool in statics and dynamics [15]. A review about the efforts in dynamics is published by Beskos [6,7]. Main advantages of the method are the reduction of the problem dimension by one and the implicit fulfilment of the radiation condition for unbounded domains.

For transient elastodynamic problems, the BEM is mostly formulated in frequency or Laplace domain followed by an inverse transformation, e.g. [1]. Mansur [27] developed one of the first boundary element formulations in the time domain for the scalar wave equation and for elastodynamics with zero initial conditions [28]. The extension of this formulation to non-zero initial conditions was presented by Antes [2]. Detailed information about this procedure can be found in the book of Domínguez [11]. Frequent

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applications are soil-structure-interaction problems (e.g., [5]), dynamic analysis of 3-d foundations [23] and contact problems [3].

Viscoelastic solids can be effectively treated by BEM in Laplace domain. However, calculation of transient response via the Laplace domain requires the inverse transformation. Since all numerical inversion formulas depend on a proper choice of their parameters [29] a direct evaluation in time domain seems to be preferable. On the other hand direct calculation of viscoelastic solids in time domain requires the knowledge of viscoelastic fundamental solutions.

Such solutions can be obtained by means of an elastic–viscoelastic correspondence principle. In Ref. [19] the elastic fundamental solution has been transformed into Laplace domain and then the elastic–viscoelastic correspondence principle has been adopted. For a simple rheological material model an analytical inverse Laplace transformation is given. This leads to the viscoelastic fundamental solution in time domain.

For the implementation of this viscoelastic fundamental solution in a 3-d time domain BEM program it is advantageous to perform the time integration analytically in a time step. This has been carried out successfully in Ref. [34], but leads to a very complicated series solution. An alternative approach to obtain a viscoelastic boundary integral formulation in time domain is treated by the authors in Ref. [17]. The generalization of this approach for generalized constitutive equations with better curve fitting properties of measured data is presented in Ref. [18]. A completely different approach on the basis of the 'convolution quadrature method' developed by Lubich [24] is published in Ref. [33].

In the present paper all three approaches, calculation in Laplace domain and the two time domain formulations, are presented. Results of forced wave propagation in a 3-d rod are compared. Finally, the wave propagation in an elastic concrete foundation slab bonded to viscoelastic semi-infinite soil is studied.

2. Constitutive equation

Decomposition of the stress tensor σ_{ij} into the the hydrostatic part $\delta_{ij}\sigma_{kk}/3$ and the deviatoric part s_{ij} yields

$$\sigma_{ij} = \tfrac{1}{3}\sigma_{kk}\delta_{ij} + s_{ij}, \quad \text{where } s_{ii} = 0. \tag{1}$$

The corresponding decomposition holds for the strain tensor ε_{ij}

$$\varepsilon_{ij} = \tfrac{1}{3}\varepsilon_{kk}\delta_{ij} + e_{ij}, \quad \text{where } e_{ii} = 0. \tag{2}$$

Two independent sets of constitutive equations for viscoelastic materials exist after this decomposition

$$\sum_{k=0}^{N} p'_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} s_{ij} = \sum_{k=0}^{M} q'_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} e_{ij}, \qquad \sum_{k=0}^{N} p''_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \sigma_{ii} = \sum_{k=0}^{M} q''_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \varepsilon_{ii}. \tag{3}$$

More flexibility in fitting measured data in a large frequency range is obtained by replacing the integer order time derivatives by fractional order time derivatives [16].

The derivative of fractional order α is defined by

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{x(t-\tau)}{\tau^\alpha} \mathrm{d}\tau \quad 0 \leqslant \alpha < 1 \tag{4}$$

with the Gamma function $\Gamma(1-\alpha) = \int_0^\infty \mathrm{e}^{-x} x^{-\alpha} \mathrm{d}x$, as the inverse operation of fractional integration attributed to Riemann and Liouville [30]. A different definition based on generalized finite differences is given by Grünwald [21]

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = \lim_{N\to\infty} \left\{ \left(\frac{t}{N}\right)^{-\alpha} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} x\left(t\left[1-\frac{j}{N}\right]\right) \right\}. \tag{5}$$

This discrete definition is more convenient in constitutive equations treated by time stepping algorithms and the equivalence of definition (4) and (5) can be shown.

The fractional derivatives in Eq. (4) appear complicated in time domain. However, Laplace transform reveals the useful result

$$\mathscr{L}\left\{\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}}\right\} = s^{\alpha}\mathscr{L}\{x(t)\} - \sum_{k=0}^{n-1} s^{k}\frac{\mathrm{d}^{\alpha-1-k}}{\mathrm{d}t^{\alpha-1-k}}x(0), \quad n-1<\alpha\leqslant n, \tag{6}$$

where s is the Laplace variable.

With the definition (4) or (5) the generalized viscoelastic constitutive equations are given by

$$\sum_{k=0}^{N} p'_k \frac{\mathrm{d}^{\alpha_k}}{\mathrm{d}t^{\alpha_k}} s_{ij} = \sum_{k=0}^{M} q'_k \frac{\mathrm{d}^{\beta_k}}{\mathrm{d}t^{\beta_k}} e_{ij}, \qquad \sum_{k=0}^{N} p''_k \frac{\mathrm{d}^{\alpha_k}}{\mathrm{d}t^{\alpha_k}} \sigma_{ii} = \sum_{k=0}^{M} q''_k \frac{\mathrm{d}^{\beta_k}}{\mathrm{d}t^{\beta_k}} \varepsilon_{ii}. \tag{7}$$

For $N = M = 1$ and fractional time derivatives of order α and β a possible representation of a uniaxial stress strain equation with 5 parameters (E Young's modulus, p, q viscoelastic parameters and the orders of the fractional derivatives α and β)

$$p\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\sigma + \sigma = E\left(\varepsilon + q\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\varepsilon\right) \tag{8}$$

is given. For wave propagation problems, e.g., in unbounded domains, the wave velocity is important. In viscoelasticity it is defined with the initial modulus $E(0)$ of the relaxation function [8]

$$c = \sqrt{\frac{E(0)}{\rho}}. \tag{9}$$

For the determination of the initial modulus, the initial value theorem of Laplace transform $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\mathscr{L}\{f(s)\}$ is used. This theorem and the definition of the initial modulus lead to

$$E(0) = \lim_{s\to\infty} E\frac{1+ps^{\alpha}}{1+qs^{\beta}} = E\lim_{s\to\infty}\frac{(1/s^{\beta})+ps^{\alpha-\beta}}{(1/s^{\beta})+q}. \tag{10}$$

The limiting process gives a finite value only for the restriction $\alpha = \beta$. For solid viscoelastic materials this restriction is in accordance with experimental data.

Finally, according to Ref. [4] a credible model of the viscoelastic phenomenon should predict nonnegative internal work and a nonnegative rate of energy dissipation. To satisfy these restrictions, constraints on the remaining parameters of the model are developed. This produces the constraints [31]

$$E > 0, \quad q > p > 0, \quad 0 < \alpha = \beta < 2. \tag{11}$$

A powerful tool for calculating viscoelastic behaviour from a known elastic response is the elastic–viscoelastic correspondence principle. According to this principle [14] the viscoelastic solution is calculated from the analytical elastic solution by replacing the elastic moduli in the Fourier or Laplace transformed domain by the transformed impact response functions of the viscoelastic material model. The viscoelastic solution is then obtained by inverse transformation.

As an example for calculating viscoelastic behaviour with the elastic–viscoelastic correspondence principle a free-fixed 1-d rod is used (Fig. 1). The solutions for the displacement $u(x,t)$ can be determined with inverse Laplace transformation from the elastic solution in Laplace domain [20]

$$\hat{u}(x,s) = \frac{\hat{F}(s)\,\sinh(sx/c)}{\rho c\;s\cosh(s\ell/c)}, \tag{12}$$

after inserting the elastic–viscoelastic correspondence

$$E \to E\frac{1+ps^{\alpha}}{1+qs^{\alpha}} \quad\Rightarrow\quad c \to c\sqrt{\frac{1+ps^{\alpha}}{1+qs^{\alpha}}}. \tag{13}$$

In the equations above $\hat{()}$ denotes the Laplace transformed functions. Whereas the elastic solution can be obtained by an analytical inversion, the viscoelastic one is performed by a numerical inverse transformation.

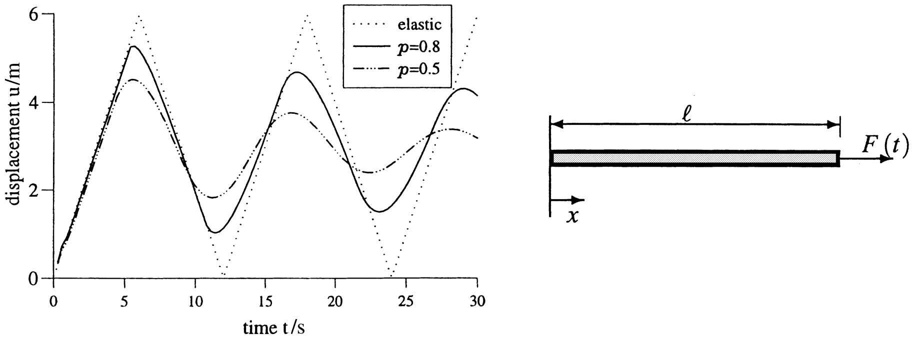


Fig. 1. Displacement at $x = \ell$ of a free-fixed 1-d rod versus time for different viscoelastic parameters p.

Here, the method of Talbot [35] is used. For $F(t) = 1\,\mathrm{N}\,H(t)$, $E = 1\ \mathrm{N/m^2}$, $\rho = 1\ \mathrm{kg/m^3}$, $q = 1\ \mathrm{s}^{-0.8}$, $\alpha = 0.8$ and $\ell = 3$ m the displacement $u(\ell, t)$ is plotted versus time for different viscoelastic parameters p in Fig. 1.

For 3-d continua the elastic–viscoelastic correspondence principle is as simple as in the 1-d case. If the same damping mechanisms are assumed in hydrostatic and deviatoric stress strain state, the corresponding 3-d constitutive equations are gained by replacing the uniaxial stress and strain by the hydrostatic and the deviatoric states. For the above-mentioned generalized three parameter model (8) the elastic–viscoelastic correspondence is the same as in the 1-d case (13) with unchanged real Poisson's ratio ν. The last condition is only valid if the same damping mechanisms are present [31].

As for the above-mentioned 1-d rod example, construction of fundamental solutions by the elastic–viscoelastic correspondence principle leads to viscoelastic solutions in Laplace or Fourier domain, firstly. Afterwards, a numerical inverse transformation is necessary.

3. Comparison of numerical inverse transformation algorithms

As only few viscoelastic constitutive equations of minor practical importance allow analytical inverse Laplace transformation, the performance of numerical inverse transformation methods is analysed below. Based on the study in Ref. [29], here, the algorithms of Durbin [13] and Crump [9] are taken into consideration and additionally the method of Talbot [35]. The two first mentioned methods are further developments of the algorithm proposed by Dubner and Abate [12]. The main idea of the method is a Fourier series expansion of the complex inversion formula of Laplace transformation. In contradiction to this the method of Talbot maps the complex integration along the Bromwich contour with infinite limits into a real integral with finite integration limits.

Table 1 compares the results of inverse transformation for the following test functions at discrete times

- function 1: $\frac{e^{-sa}}{s} \bullet\!\!-\!\!\circ H(t-a)$
- function 2 : $\frac{1}{\sqrt{s(s+a)}} e^{-\frac{r}{c}\sqrt{s(s+\alpha)}} \bullet\!\!-\!\!\circ I_0\left(\frac{\alpha}{2}\sqrt{t^2-\left(\frac{r}{c}\right)^2}\right) e^{-\frac{\alpha t}{2}} H\left(t-\frac{r}{c}\right)$
- function 3: $\frac{1}{s}\, e^{-s\frac{r}{c}\sqrt{\frac{1+ps^{0.7}}{1+qs^{0.7}}}} \bullet\!\!-\!\!\circ$ no analytical solution

where $H(\,)$ denotes the Heaviside function and $a = 1, r/c = 0.5, p = 1$ and $q = 2$. The dashes in the table denotes no solution or nonconverging method. The test functions were selected because of the following reasons:

- shifting of the Heaviside function in function 1 occurs in the elastodynamic fundamental solution (23) and represents the wave front, which is also expected in the viscoelastic case,
- function 2 is part of the transformed fundamental solution with viscoelastic Maxwell model [34],
- function 3 is part of Eq. (20) after applying the elastic–viscoelastic correspondence principle (13).

The results in Table 1 for function 1 correspond to times before and after the Heaviside jump at $t = a$. The method of Talbot fails for $t < a$. Only the results of the method by Crump can be considered zero numerically in this range.

Table 1
Comparison of inverse transformation algorithms

	Time (s)	Talbot	Exact	Durbin	Crump
	0.7	-3.91×10^{50}	0.000000	7.88×10^{-4}	1.60×10^{-9}
	0.9	-5.04×10^{12}	0.000000	9.67×10^{-4}	1.95×10^{-8}
function 1	1.1	1.000000	1.000000	0.998886	0.999999
	1.3	1.000000	1.000000	0.998565	1.000000
	1.5	1.000000	1.000000	1.004410	1.000000
	0.5	0.394691	0.389400	0.389363	–
	0.6	0.708395	0.745920	0.746377	0.745920
function 2	0.7	0.715297	0.715298	0.716224	0.715298
	0.8	0.686752	0.686759	0.690466	0.686759
	0.9	0.660142	0.660141	0.663267	0.660141
	0.5	0.906345	–	0.906504	–
	0.6	0.926606	–	0.925405	0.926606
function 3	0.7	0.938730	–	0.942046	0.938730
	0.8	0.947151	–	0.943823	0.947151
	0.9	0.953470	–	0.949619	0.953470

Based on a more detailed comparison [31] it can be concluded, that the methods by Talbot and Crump lead to better accuracy than the method by Durbin, which is as well more time consuming for complicated functions. The method of Crump seams to be very suitable, but it is obviously not so robust like the method of Durbin, i.e., for some times it does not converge. If jumps can be avoided the method of Talbot should be preferred. Nevertheless, for all three methods a proper choice of the parameters is essential.

4. Viscoelastic BE-formulations

Assuming homogeneity and linear viscoelastic material the dynamic behaviour of a waveguide as well as the wave propagation in an unbounded domain Ω will be determined. In the following, Latin indices take the values 1,2, and 1,2,3 in 2-d and 3-d, respectively, where summation convention is implied over repeated indices, and commas and over-dots denote spatial and time differentiation, respectively. On the boundary $\Gamma = \Gamma_p \cup \Gamma_u$ of the domain Ω, tractions $p_i(\mathbf{x}, t)$ and displacements $u_j(\mathbf{x}, t)$, respectively, i.e.

$$\sigma_{ij} n_j = p_i(\mathbf{x}, t) \quad \text{for } t > 0, \ \mathbf{x} \in \Gamma_p, \qquad u_i(\mathbf{x}, t) = q_i(\mathbf{x}, t) \quad \text{for } t > 0, \ \mathbf{x} \in \Gamma_u \tag{14}$$

are prescribed, while the initial conditions $u_i(\mathbf{x}, 0)$ and $\dot{u}_i(\mathbf{x}, 0), \mathbf{x} \in \Omega \cup \Gamma$ are assumed to be zero

$$u_i(\mathbf{x}, 0) = 0, \qquad \dot{u}_i(\mathbf{x}, 0) = 0 \quad \mathbf{x} \in \Omega \cup \Gamma. \tag{15}$$

σ_{ij} is the stress tensor and n_j means the outward normal vector on the boundary Γ. Neglecting the body force effects, the dynamic extension of Betti's reciprocal work theorem combining two states of displacements and tractions (U_{ij}, P_{ij}) and (u_j, p_j) leads to the integral equation [31]

$$c_{ij}(\mathbf{y}) u_j(\mathbf{y}, t) = \int_{\Gamma_x} U_{ij}(\mathbf{x}, \mathbf{y}, t) * p_j(\mathbf{x}, t) \mathrm{d}\Gamma_x - \oint_{\Gamma_x} P_{ij}(\mathbf{x}, \mathbf{y}, t) * u_j(\mathbf{x}, t) \mathrm{d}\Gamma_x, \tag{16}$$

where $*$ denotes the convolution with respect to time. U_{ij} and P_{ij} are the displacements and tractions, respectively, due to a unit impulse in the direction x_i, i.e., the time-dependent fundamental solution of the full space. For the linear viscoelastic material described in Section 2, e.g., the generalized 3-parameter model (8), the integral free terms $c_{ij}(\mathbf{y})$ are identical to those of elastostatics and dependent only on the local geometry at $\mathbf{y}$ and on Poisson's ratio ν, i.e., for $\mathbf{y}$ on a smooth boundary $c_{ij}(\mathbf{y}) = \delta_{ij}/2$. However, this is valid only for the assumption of the same viscoelastic behaviour of the deviatoric and hydrostatic stress strain state.

When $\mathbf{x}$ approaches $\mathbf{y}$, the kernel $U_{ij}(\mathbf{x},\mathbf{y},t)$ is weakly singular and $P_{ij}(\mathbf{x},\mathbf{y},t)$ is strongly singular, i.e., the second integral in Eq. (16) exists only in the Cauchy principal value sense.

According to the boundary element method the boundary surface Γ is discretized by E iso-parametric elements Γ_e where F polynomial spatial shape functions $N_e^f(\mathbf{x})$ are defined. Hence, with the time dependent nodal values $u_j^{ef}(t)$ and $p_j^{ef}(t)$ the following representation is adapted

$$u_j(\mathbf{x},t)=\sum_{e=1}^{E}\sum_{f=1}^{F}N_e^f(\mathbf{x})u_j^{ef}(t),\quad p_j(\mathbf{x},t)=\sum_{e=1}^{E}\sum_{f=1}^{F}N_e^f(\mathbf{x})p_j^{ef}(t). \tag{17}$$

Inserting these 'ansatz' functions in Eq. (16) gives

$$c_{ij}(\mathbf{y})u_j(\mathbf{y},t)=\sum_{e=1}^{E}\sum_{f=1}^{F}\left\{\int_{\Gamma_e}U_{ij}(\mathbf{x},\mathbf{y},t)N_e^f(\mathbf{x})\mathrm{d}\Gamma_x * p_j^{ef}(t)-\oint_{\Gamma_e}P_{ij}(\mathbf{x},\mathbf{y},t)N_e^f(\mathbf{x})\mathrm{d}\Gamma_x * u_j^{ef}(t)\right\}. \tag{18}$$

In what follows, usually, the fundamental solutions must be known. Only for the simplest viscoelastic model, the Maxwell material, the fundamental solutions are available analytically [17]. For more realistic models the fundamental solutions are only known in Laplace domain. Until now, three different boundary element formulations have been proposed. These are summarized here and afterwards described in more detail:

- Formulation 1: Uses the boundary element method in Laplace domain and a subsequent numerical inverse transformation, proposed, e.g., by [1].
- Formulation 2: Time shape functions for the unknown displacements and tractions are chosen and the convolution of these functions with the elastodynamic fundamental solution is integrated analytically. The result is transformed in Laplace domain, where the elastic–viscoelastic correspondence principle is applied. A numerical inverse transformation leads to a time domain boundary element formulation [18].
- Formulation 3: Uses the so called 'convolution quadrature method' developed by Lubich [24] to evaluate the convolution in Eq. (18). This method only requires the fundamental solutions inLaplace domain [33].

In all three formulations the spatial integration of the regular integrals over the boundary elements is performed by Gaussian quadrature. The weakly singular integrals are regularized with polar coordinate transformation and the strongly singular integrals with the method proposed by Guiggiani [22]. Finally, the solution of the resulting algebraic system of equations is solved by a direct solver.

Formulation 1: The Laplace transform of the integral equation (18) is

$$c_{ij}(\mathbf{y})\hat{u}_j(\mathbf{y},s)=\sum_{e=1}^{E}\sum_{f=1}^{F}\left\{\int_{\Gamma_e}\hat{U}_{ij}(\mathbf{x},\mathbf{y},s)N_e^f(\mathbf{x})\hat{p}_j^{ef}(s)\mathrm{d}\Gamma_x-\oint_{\Gamma_e}\hat{P}_{ij}(\mathbf{x},\mathbf{y},s)N_e^f(\mathbf{x})\hat{u}_j^{ef}(s)\mathrm{d}\Gamma_x\right\}. \tag{19}$$

The viscoelastic fundamental solutions $\hat{U}_{ij}(\mathbf{x},\mathbf{y},s)$ and $\hat{P}_{ij}(\mathbf{x},\mathbf{y},s)$ are obtained by applying the elastic–viscoelastic correspondence to the elastodynamic solutions [10]

$$\hat{U}_{ij}(\mathbf{x},\mathbf{y},s)=\frac{1}{4\pi\rho}\left\{\frac{1}{r^2}\left(\frac{3r_ir_j}{r^3}-\frac{\delta_{ij}}{r}\right)\left[\frac{s\frac{r}{c_1}+1}{s^2}\mathrm{e}^{-\frac{r}{c_1}s}-\frac{s\frac{r}{c_2}+1}{s^2}\mathrm{e}^{-\frac{r}{c_2}s}\right]+\frac{r_ir_j}{r^3}\left[\frac{1}{c_1^2}\mathrm{e}^{-\frac{r}{c_1}s}-\frac{1}{c_2^2}\mathrm{e}^{-\frac{r}{c_2}s}\right]+\frac{\delta_{ij}}{rc_2^2}\mathrm{e}^{\frac{r}{c_2}s}\right\}, \tag{20}$$

where $r=\sqrt{r_ir_i}$ with $r_i=x_i-y_i$. The corresponding fundamental traction components needed in Eq. (19) are obtained via

$$\hat{P}_{ij}=\rho(c_1^2-2c_2^2)\hat{U}_{jm,m}\delta_{ik}n_k+\rho c_2^2(\hat{U}_{ji,k}n_k+\hat{U}_{jk,i}n_k) \tag{21}$$

with the Kronecker symbol δ_{ik}. The correspondence principle simply replaces the elastic wave velocities by the viscoelastic ones

$$c_1=\sqrt{\frac{E(1-\nu)}{\varrho(1-2\nu)(1+\nu)}}\rightarrow c_1\sqrt{\frac{1+qs^\alpha}{1+ps^\alpha}},\quad c_2=\sqrt{\frac{E}{\varrho 2(1+\nu)}}\rightarrow c_2\sqrt{\frac{1+qs^\alpha}{1+ps^\alpha}} \tag{22}$$

in Eqs. (20) and (21). As next step, the solution is transformed back to time domain. Preferable methods for this application are those of Durbin or Crump, according to the experience reported in [26].

Formulation 2: As has been shortly summarized above, this formulation uses the elastodynamic fundamental solutions in time domain, e.g., for the displacements

$$U_{ij}(\mathbf{x},\boldsymbol{\xi},t,\tau)=\frac{1}{4\pi\rho}\left\{(t-\tau)f_0(r)\left[H\left(t-\tau-\frac{r}{c_1}\right)-H\left(t-\tau-\frac{r}{c_2}\right)\right]\right. \\ \left.+\frac{f_1(r)}{c_1^2}\delta\left(t-\tau-\frac{r}{c_1}\right)+\frac{f_2(r)}{c_2^2}\delta\left(t-\tau-\frac{r}{c_2}\right)\right\} \tag{23}$$

with abbreviated functions depending on spatial coordinates only

$$f_0(r)=\frac{3r_{,i}r_{,j}-\delta_{ij}}{r^3},\quad f_1(r)=\frac{r_{,i}r_{,j}}{r},\quad f_2(r)=\frac{\delta_{ij}-r_{,i}r_{,j}}{r}. \tag{24}$$

The solution for the tractions is calculated by the time domain expression corresponding to Eq. (21). The elapsed time t is discretized by N equal time-increments Δt. The simplest nontrivial choice for the time shape functions, ensuring that no terms drop out in the boundary integral equation after one differentiation, are linear shape functions for the displacements $u_j^{ef}(t)$ and constant shape functions for the tractions $p_j^{ef}(t)$

$$u_j^{ef}(t)=\left(u_{j(m-1)}^{ef}\frac{t_m-\tau}{\Delta t}+u_{jm}^{ef}\frac{\tau-t_{m-1}}{\Delta t}\right),\qquad p_j^{ef}(t)=1\cdot p_{jm}^{ef} \tag{25}$$

The actual time step is m. Inserting these ‘ansatz’ function in Eq. (18) analytical time integration can be carried out within each time step because of the properties of the Dirac and Heaviside functions. This integration leads to piecewise defined functions. For the sake of brevity the integration is indicated only for the first term on the right side of Eq. 18)

$$\int_{t_{m-1}}^{t_m}U_{ij}(\mathbf{x},\boldsymbol{\xi},t-\tau)\mathrm{d}\tau=\frac{1}{4\pi\rho}\begin{cases}0, & t<t_{m-1}+\frac{r}{c_1},\\ \frac{f_0(r)}{2}\left[(t-t_{m-1})^2-(\frac{r}{c_1})^2\right]+\frac{f_1(r)}{c_1^2}, & t_{m-1}+\frac{r}{c_1}<t<t_m+\frac{r}{c_1},\\ f_0(r)\left[tt_m-tt_{m-1}-\frac{t_m^2}{2}+\frac{t_{m-1}^2}{2}\right], & t_m+\frac{r}{c_1}<t<t_{m-1}+\frac{r}{c_2},\\ \frac{f_0(r)}{2}\left[(\frac{r}{c_2})^2-(t_m-t)^2\right]+\frac{f_2(r)}{c_2^2}, & t_{m-1}+\frac{r}{c_2}<t<t_m+\frac{r}{c_2},\\ 0, & t_m+\frac{r}{c_2}<t.\end{cases} \tag{26}$$

Now, after performing the convolution, function (26) and the corresponding function for the tractions are transformed in the Laplace domain

$$\int_0^\infty\sum_{m=1}^n\int_{t_{m-1}}^{t_m}U_{ij}(\mathbf{x},\boldsymbol{\xi},t-\tau)\ \mathrm{d}\tau\ \mathrm{e}^{-st}\mathrm{d}t=\frac{1}{4\pi\varrho}\sum_{m=1}^n\left\{\int_{t_{m-1}+\frac{r}{c_1}}^{t_m+\frac{r}{c_1}}\left[\frac{f_0(r)}{2}\left((t-t_{m-1})^2-(\frac{r}{c_1})^2\right)+f_1(r)\right]\mathrm{e}^{-st}\mathrm{d}t\right. \\ +\int_{t_m+\frac{r}{c_1}}^{t_{m-1}+\frac{r}{c_2}}f_0(r)\left(t\,t_m-t\,t_{m-1}-\frac{t_m^2}{2}+\frac{t_{m-1}^2}{2}\right)\mathrm{e}^{-st}\mathrm{d}t \\ \left.+\int_{t_{m-1}+\frac{r}{c_2}}^{t_m+\frac{r}{c_2}}\left[\frac{f_0(r)}{2}\left(\frac{r}{c_2}\right)^2-(t-t_m)^2+f_2(r)\right]\mathrm{e}^{-st}\ \mathrm{d}t\right\} \\ =\frac{1}{4\pi\varrho}\sum_{m=1}^n(\mathrm{e}^{-st_{m-1}}-\mathrm{e}^{-st_m})\left\{f_0(r)\frac{1}{s^2}\left[\left(\frac{r}{c_1}\right)\mathrm{e}^{-\frac{r}{c_1}s}-\left(\frac{r}{c_2}\right)\mathrm{e}^{-rc_2s}\right]\right. \\ \left.+f_1(r)\frac{1}{s}\mathrm{e}^{-rc_1s}+f_2(r)\frac{1}{s}\mathrm{e}^{-\frac{r}{c_2}s}\right\}. \tag{27}$$

In Laplace domain the elastic–viscoelastic correspondence (22) is applied. Causality of the solution implies that no response is present prior to the arrival of the waves. This physical requirement is assumed for the numerical inversion of Eq. (27). The correspondence relation

$$\mathscr{L}f\mathrm{e}^{-st_{m-1}}-\mathscr{L}f\mathrm{e}^{-st_m}\bullet\!\!-\!\!\circ f(t-t_{m-1})H(t-t_{m-1})-f(t-t_m)H(t-t_m) \tag{28}$$

is taken into account as well. The inverse transformation therefore leads to a function defined piecewise in the time ranges of Eq. (26) with the viscoelastic wave velocities

$$c_1^{\text{viscoelastic}} = c_1\sqrt{\frac{q}{p}} \qquad c_2^{\text{viscoelastic}} = c_2\sqrt{\frac{q}{p}}, \tag{29}$$

defined in accordance to Eqs. (9) and (10). The inversion is carried out numerically by the method of Talbot, because the method of Crump produces convergence problems and the method of Durbin is very time consuming.

Formulation 3: In this formulation the convolution between the fundamental solutions and the corresponding nodal values in Eq. (18) is performed numerically with the so-called 'convolution quadrature method' proposed by Lubich [24,25]. The quadrature formula approximates a convolution integral

$$y(t) = f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau \tag{30}$$

by the finite sum

$$y(n\Delta t) = \sum_{j=0}^{n} \omega_{n-j}(\Delta t) g(j\Delta t), \quad n = 0, 1, \ldots, N. \tag{31}$$

The integration weights $\omega_n(\Delta t)$ are the coefficients of the power series for the function $\hat{f}(\Gamma(z)/\Delta t$ at the point $\gamma(z)/\Delta t$. Herein, $\gamma(z)$ is the quotient of the characteristic polynomials of a linear multistep method, e.g., the backward differentiation formula of second order $\gamma(z) = 3/2 - 2z + (1/2)z^2$. The coefficients are calculated by the integral

$$\omega_n(\Delta t) = \frac{1}{2\pi i}\int_{|z|=\imath} \hat{f}\left(\frac{\gamma(z)}{\Delta t}\right) z^{-n-1}\mathrm{d}z \approx \frac{\imath^{-n}}{L}\sum_{\ell=0}^{L-1} \hat{f}\left(\frac{\gamma(\imath \mathrm{e}^{i\ell\frac{2\pi}{L}})}{\Delta t}\right)\mathrm{e}^{-in\ell\frac{2\pi}{L}}, \tag{32}$$

with $\imath$ being the radius of a circle in the domain of analyticity of $\hat{f}(z)$. The integral in Eq. (32) is approximated by a trapezoidal rule with L equal steps $2\pi/L$, after transformation to polar coordinates. Details of the convolution quadrature method can be found in [24,25]. An example of this quadrature formula related to the boundary element method is presented in [32].

The quadrature formula (31) is applied to Eq. (18). The result is the following boundary element time-stepping formulation for $n = 0, 1, \ldots, N$

$$c_{ij}(\mathbf{y})u_j(\mathbf{y}, n\Delta t) = \sum_{e=1}^{E}\sum_{f=1}^{F}\sum_{k=0}^{n}\left\{\omega_{n-k}^{ef}(\hat{U}_{ij}, \mathbf{y}, \Delta t)p_j^{ef}(k\Delta t) - \omega_{n-k}^{ef}(\hat{P}_{ij}, \mathbf{y}, \Delta t)u_j^{ef}(k\Delta t)\right\} \tag{33}$$

with the weights corresponding to Eq. (32)

$$\omega_n^{ef}(\hat{U}_{ij}, \mathbf{y}, \Delta t) = \frac{\imath^{-n}}{L}\sum_{l=0}^{L-1}\int_{\Gamma_e} \hat{U}_{ij}\left(\mathbf{x}, \mathbf{y}, \frac{\gamma\left(\imath \mathrm{e}^{il\frac{2\pi}{L}}\right)}{\Delta t}\right) N_e^f(\mathbf{x})\mathrm{d}\Gamma_x \mathrm{e}^{-in\ell\frac{2\pi}{L}}. \tag{34}$$

$$\imath\omega_n^{ef}(\hat{P}_{ij}, \mathbf{y}, \Delta t) = \frac{\imath^{-n}}{L}\sum_{l=0}^{L-1}\oint_{\Gamma_e} \hat{P}_{ij}\left(\mathbf{x}, \mathbf{y}, \frac{\gamma\left(\imath \mathrm{e}^{il\frac{2\pi}{L}}\right)}{\Delta t}\right) N_e^f(\mathbf{x})\mathrm{d}\Gamma_x \mathrm{e}^{-in\ell\frac{2\pi}{L}}. \tag{35}$$

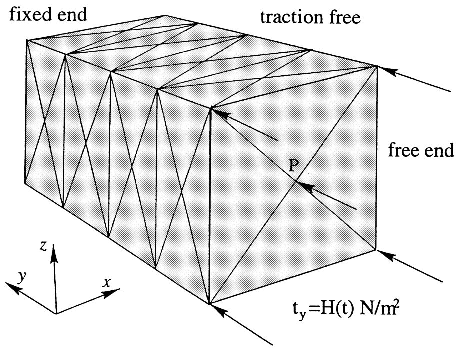
Note that the calculation of the quadrature weights (34) and (35) is only based on the Laplace transformed fundamental solutions. Therefore, applying the elastic–viscoelastic correspondence principle to the fundamental solutions (20) and (21) leads to a visco-elastodynamic boundary element formulation in time domain. The calculation of the integration weights (34) and (35) is performed very fast with a technique similar to the Fast Fourier Transform (FFT).

5. Numerical examples

The propagation of waves in a 3-d continuum has been calculated by the presented three viscoelastic boundary element formulations. The first part of the examples compare numerical results of the three methods. As well, the wave propagation in a 3-d rod is compared with the 1-d solution. In the second part, the wave propagation in an elastic foundation slab bonded to a viscoelastic half-space is calculated.

5.1. Comparison of the methods

The problem geometry, material data and the associated boundary discretization of the 3-d rod are shown in Fig. 2. The rod is fixed on one end, and is excited by a pressure jump according to a unit step function $H(t)$ on the other free end. The remaining surfaces are traction free. Linear spatial shape functions on 72 triangles are used. Fig. 3 shows the longitudinal displacement in the centre of the free end cross section (point P) versus time for all three presented formulations and the 1-d solution. The difference between the results, also compared to the 1-d solution, is quite small. The time step size Δt is chosen optimal for formulation 2 and 3, respectively. This means for formulation 2 the value $\beta = c_1 \Delta t / r_e$, with the characteristic element-length r_e, is in the range $0.6 < \beta < 1$ [31] and for formulation 3 is $\beta \approx 0.15$. However, formulation 3 is not as sensitive to an optimal choice of β as formulation 2. This means, if e.g. $\beta = 0.6$ is



material data:			geometry data:		
E	=	$2.075 \cdot 10^6 \frac{N}{m^2}$	length	=	4 m
ν	=	0	height	=	2 m
ρ	=	$1000 \frac{kg}{m^3}$	width	=	2 m
α	=	1			
q	=	$5\,s^{-1}$			
p	=	$3.5\,s^{-1}$			

Fig. 2. Step function excitation of a free-fixed rod.

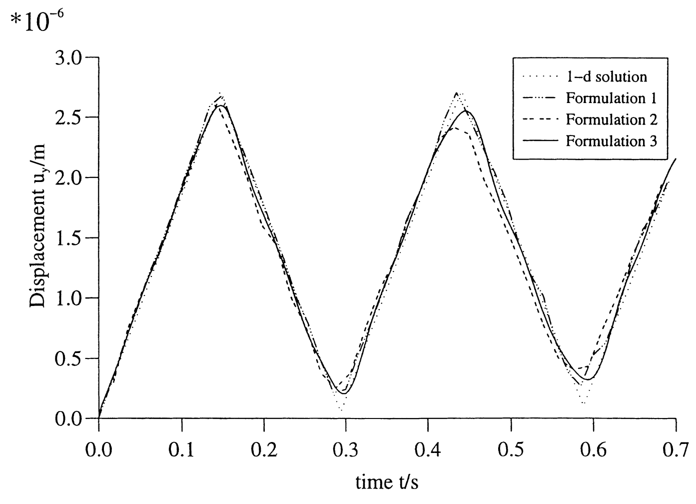


Fig. 3. Longitudinal displacement of point P.

used in formulation 3 the results differ not much from the results with $\beta = 0.15$. Below critical values, $\beta = 0.6$ in formulation 2 and $\beta = 0.1$ in formulation 3, both formulations become unstable.

A proper parameter choice is also necessary for formulation 1. In this formulation the real part of the selected Laplace variable s must be tuned such that the solution behaves well. A bad choice of this parameter can lead to wrong results. Furthermore, this parameter is problem dependant, so that no general rule can be given how to choose this parameter.

From the viewpoint of CPU-time formulation 3 must be preferred, because it is nearly ten times faster than formulation 2. An comparison of CPU-time with the Laplace domain method is not general valid, because the amount of CPU-time depends linearly on the used amount of frequencies. However, this amount is closely related to the problem.

Summarizing the problems of all three formulations, the conclusion of a comparison is that formulation 3 is a very effective formulation and simple as well to treat viscoelastic problems. Nevertheless, both other formulations can be used as well, but their parameters must be chosen very carefully.

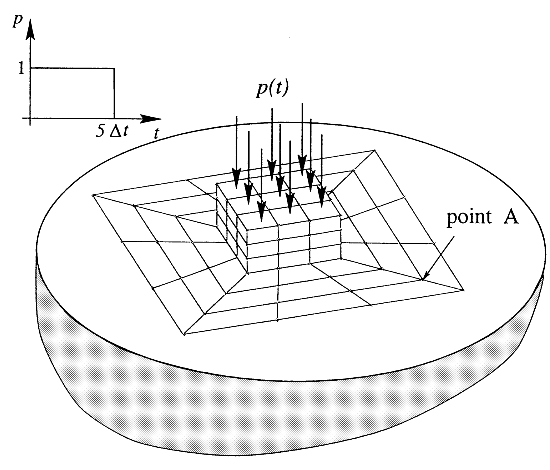
5.2. Wave propagation in an unbounded half-space

The propagation of waves in an elastic concrete foundation slab ($2m \times 2m \times 1m$) bonded on a viscoelastic soil half-space has been calculated by formulation 2. Both domains are coupled by a substructure technique based on displacement- and traction-continuity at the interface. The assumption of welded contact does not allow the nonlinear effect of partial uplifting.

The problem geometry and the associated boundary discretization are shown in Fig. 4. The soil discretization is truncated after a distance of the foundation length. The surface of the foundation slab is excited by a positive and negative pressure jump according to Fig. 4. Linear spatial shape functions have been used. Similar to the Courant criterion the time step size Δt has been chosen close to the time needed by the viscoelastic compression wave to travel across the largest element.

In Fig. 5 vertical surface displacement at point A on the half-space surface is plotted versus time for different values of the constitutive parameter q. Obviously the wave speeds increase for higher values of q, because of a stiffening of the material with growing influence of viscosity. This is associated with a significant displacement reduction.

The propagation of waves on the surface of the foundation and the half-space is presented Fig. 6, where the magnitude of the displacement field at the nodes is shown. Clearly the two wave fronts of the compression and the shear wave are observed. The wave front of the compression wave can be observed during transition from the foundation slab into the half-space. The wave front of the slower wave belongs to the shear wave.



sand:

$E = 1.38 \cdot 10^8 \frac{N}{m^2}$
$\nu = 0.35$
$\rho = 1966 \frac{kg}{m^3}$
$\alpha = 1.3$
$p = 1 s^{-1.3}$
$q = 1.5/5/7.5/10 s^{-1.3}$

concrete:

$E = 3 \cdot 10^8 \frac{N}{m^2}$
$\nu = 0.20$
$\rho = 2000 \frac{kg}{m^3}$

Fig. 4. Elastic concrete slab on viscoelastic half-space: boundary element discretization, material data and loading function.

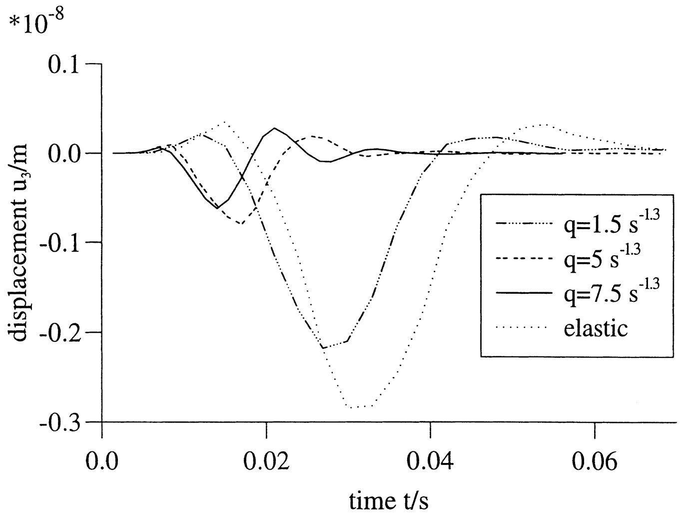


Fig. 5. Displacement u_3 perpendicular to the surface at point A for different values q.

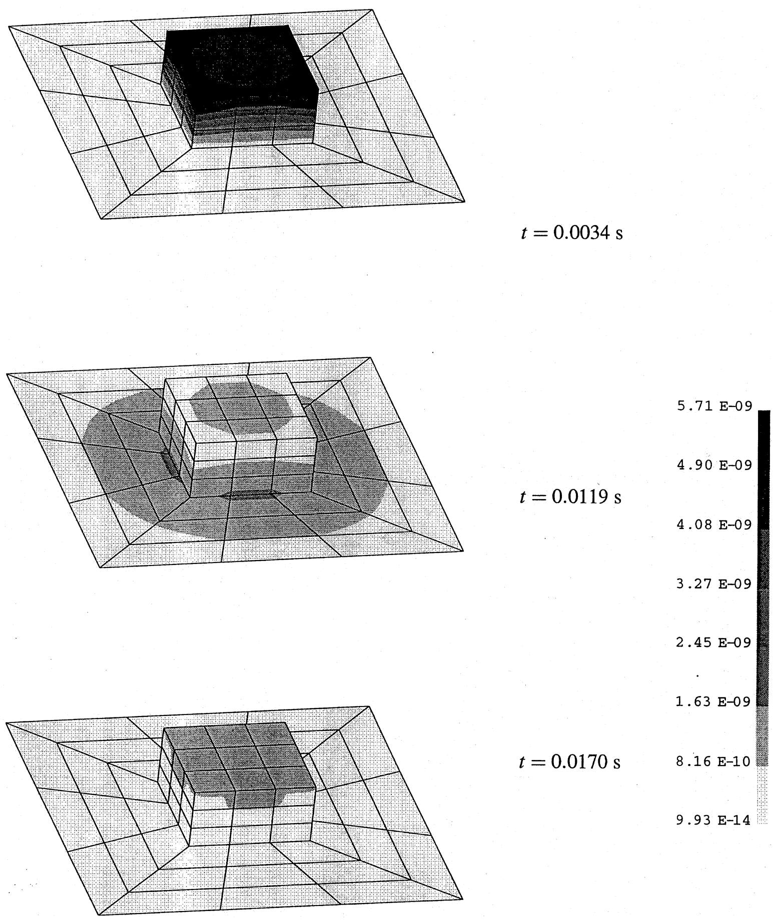


Fig. 6. Displacement field of propagating waves on the surfaces of foundation slab and half-space.

6. Conclusions

The present paper describes three BE methodologies to treat visco-elastodynamic problems in time domain. The first and oldest approach is the calculation in Laplace domain. It requires an inverse transformation if the time dependent results are of interest. The second formulation, proposed here, converts the elastic formulation in time domain to a viscoelastic one. It turned out to be advantageous to convert after integration over a time step Δt rather than converting the fundamental solution itself. The conversion is performed by applying the elastic–viscoelastic correspondence principle in Laplace domain. The inverse transformation in each time step is carried out numerically. The third approach uses the convolution quadrature formulae developed by Lubich. By application of these quadrature formulae to the convolution integral in the boundary integral equation, a numerical integration formula is obtained where the integration weights depend only on the Laplace transformed fundamental solutions and a linear multistep method. All three formulations use the Laplace transformed fundamental solutions.

The comparison of the results for the wave propagation in a 3-d rod shows good agreement between all three formulations. For the formulation in Laplace domain a proper choice of the real part of the complex Laplace variable is essential for the quality of the results [31]. The crucial parameter for the other two methods is a proper choice of the time step size, whereas the formulation based on the convolution quadrature is not as sensitive to an optimal time step size.

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