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ELEMENTARY CONSTRUCTIONS IN THE HYPERBOLIC PLANE

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ABSTRACT: Constructions of regular *n*-gons in the Poincaré disk model and in the Beltrami-Klein model of the hyperbolic geometry are presented. The focus is on methods that can be carried out by hand using only a Euclidean compass, a Euclidean protractor and a straightedge (i.e. a ruler without measuring marks). A regular *n*-gon can be decomposed into *2n* right-angled triangles. This is why we can reduce the construction of a regular *n*-gon to the construction of a right-angled triangle.

Keywords: Elementary Euclidean Geometry, Hyperbolic Plane, Beltrami-Klein Model, Poincaré Model, Geometry of Circles.

1. INTRODUCTION

Hyperbolic geometry and especially its constructive aspects have recently been reconsidered (see [1], [5] and [9]). One reason may be the fact that some computer programmes like *Cinderella* [8] provide constructions of triangles, regular polygons and even tilings in the hyperbolic plane with a few mouse clicks.

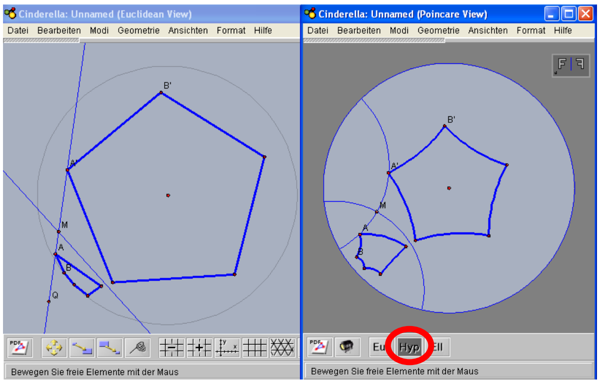

Figure 1: Regular pentagons in the Beltrami-Klein model and the Poincaré disk

In *Cinderella* we can choose both the Beltrami-Klein model and the Poincaré disk model to visualize the result (Figure 1). NonEuclid [12] is another programme to visualize hyperbolic geometry in the Poincaré disk model and in the half plane model. The aim of this work is to demonstrate the possibility to visualize hyperbolic geometry with constructions made "by hand". As an example we construct regular *n*-gons in the Poincaré disk model and in the Beltrami-Klein model applying well-known constructions of elementary Euclidean geometry.

In section 2 short summaries about a few properties of the Poincaré disk and the Beltrami-Klein model are given which will facilitate our constructions. In section 3 properties of regular *n*-gons are collected. Even though we only need to know how to construct a right-angled triangle given by two acute angles with $\alpha+\beta<\pi/2$, we will construct in section 4 a triangle with three arbitrary angles with $\alpha+\beta+\gamma<\pi/2$ in the Poincaré disk. In section 5 we construct an *n*-gon with given angle between adjacent edges in the Poincaré disk and in section 6 an *n*-gon with given side length in the Beltrami-Klein model. Section 7 is a short conclusion.

2. MODELS OF THE HYPERBOLIC PLANE

As there are a lot of textbooks about hyperbolic geometry, e.g. [3], [4], [6] and [7], only some basic notions and properties are listed here.

2.1 Poincaré disk model

Let m be a Euclidean circle. The Poincaré disk model of hyperbolic geometry is the open disk in the Euclidean plane with boundary m. Points of the model are inner points of m. Hyperbolic lines in this model are open arcs on circles orthogonal to m. Hyperbolic reflexions are reflexions across diameters of m or inversions with respect to circles orthogonal to m. (These circles c are also characterized as elements of pencils of Euclidean circles with basic points B and B^*, inverse with respect to m.) The incidence is induced from the underlying Euclidean plane. The angle measurement is identical to the Euclidean angle measurement, i.e. the Poincaré disk is a conformal model of the hyperbolic plane.

2.2 Beltrami-Klein model

The Beltrami-Klein model of hyperbolic geometry is the open disk in the Euclidean plane with boundary m. Points of the model are inner points of m. Hyperbolic lines are open cords of m. The incidence is induced from the underlying Euclidean plane. The boundary m of the disk is the so-called absolute circle. The hyperbolic orthogonality is determined by the polarity of m. A hyperbolic right angle is a Euclidean right angle iff one leg is a diameter of m. Automorphic collineations of m are hyperbolic displacements. Especially, harmonic automorphic collineations are hyperbolic reflexions. Therefore, center and axis of a hyperbolic reflexion are pole and polar of the absolute circle m. For angle measurement we will apply that the Euclidean metric and the hyperbolic metric in the center of the absolute circle m are identical and we can measure hyperbolic angles with a Euclidean protractor.

2.3 Remarks on measurement

The congruence of segments is equivalent to the statement that the segments have identical length. A similar remark applies to angles. Hence, we can specify the hyperbolic length $l = \overline{T_1T_2}^h$ by two points T_1, T_2 in the Poincaré disk and in the Beltrami-Klein model. The hyperbolic measure of an angle $\alpha = \angle^h t_1t_2$ in the Beltrami-Klein is determined by two lines t_1, t_2.

3. PROPERTIES OF REGULAR *N*-GONS IN THE HYPERBOLIC GEOMETRY

The Euclidean geometry and the hyperbolic geometry are closely related to each other. Therefore, regular n-gons share a couple of properties, e.g.: $A_1...A_n$ is regular if all its angles are congruent and all its edges have the same length with respect to the chosen metric. All the vertices of $A_1...A_n$ lie on a common circle and its center C is the center of the regular n-gon, too. Simple regular n-gons are always convex. The symmetry group of a regular n-gon is the dihedral group D_n of order $2n$. It consists of the n rotations with center C and angle $2\pi/n$, together with n reflexions with axes through the center. If n is even, then half of these axes pass through the midpoints of opposite edges. If n is odd, then all axes pass through a vertex and the midpoint of the opposite edge. No matter, if n is even or odd, two axes with angle π/n determine two congruent right-angled triangles. The vertices of one triangle are one vertex of the n-gon, the midpoint of an adjacent edge and the center of the regular n-gon. If we start with one such triangle (see Figure 4 and Figure 5), the orbit under the dihedral group is the regular n-gon. Therefore, it is clear that the construction of a right-angled triangle is the essential part of the construction of a regular n-gon. But there are no similarities in the hyperbolic geometry! Two regular n-gons $A_1...A_n$ and $B_1...B_n$ with different edge length have different angles between two adjacent edges. Therefore, a regular n-gon in the hyperbolic geometry can be given either by the angle $\sigma < \pi(n-2)/n$ between two adjacent edges (*case I*) or by the length s of an edge (*case II*).

4. CONSTRUCTION OF A TRIANGLE WITH THREE SPECIFIED ANGLES IN THE POINCARÉ DISK

4.1 A Euclidean circles intersecting two lines under specified angles

Let b and c be two intersecting lines with $\angle bc = \alpha$. To construct a Euclidean circle a^* with $\angle a^*b = \gamma$ and $\angle a^*c = \beta$ we choose some arbitrary points on b and c and draw the angles γ and β, respectively (Figure 2).

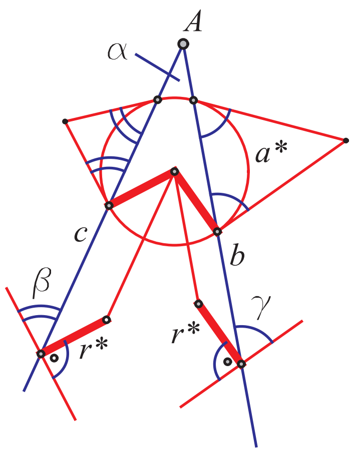

Figure 2: Euclidean circles intersect two intersecting lines under specified angles

An arbitrary distance r^* is marked on lines orthogonal to the second legs of γ and β, respectively. Parallel lines to b and c through the new points intersect in the center of a circle a^* with the desired properties.

4.2 Triangles with three specified angles in the Poincaré disk

Let m be the absolute circle of the Poincaré disk, α, β and γ be angles with $\alpha + \beta + \gamma < \pi$. Then triangles ABC with angles α, β and γ exist (they are congruent). Let us construct one. Without loss of generality we put the vertex A by a hyperbolic displacement in the center of m. The edges b and c through A are segments on diameters of m. The missing edge a of the triangle is an arc on a Euclidean circle orthogonal to m, intersecting b and c under $\angle ab = \gamma$ and $\angle ac = \beta$. Due to subsection 4.1, we can draw a circle a^* with $\angle a^*b = \gamma$ and $\angle a^*c = \beta$. In order to find a circle m^* with center A and orthogonal to a^*, we draw the polar of A with respect to a^*. The points of intersection with a^* determine the radius of m^*. The dilation with center A that maps m^* on m also maps the circle a^* on a circle a orthogonal to m (Figure 3).

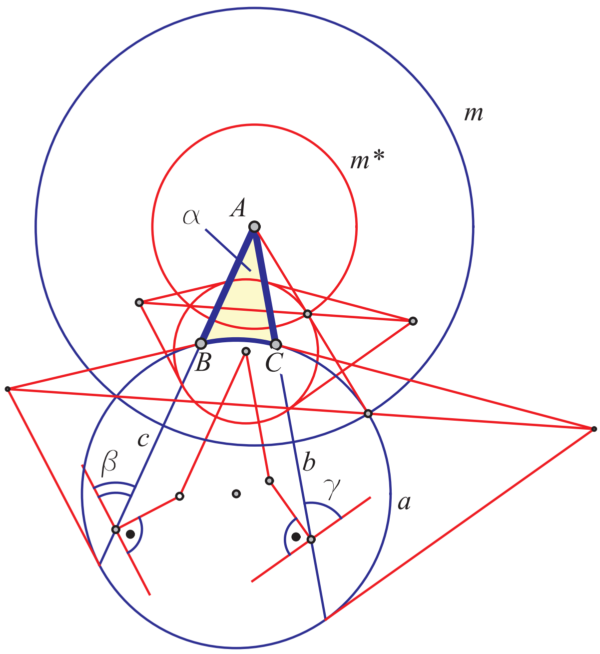

Figure 3: Triangle ABC with three specified angles and with vertex A in the center of the absolute circle of the Poincaré disk

The arc BC on the circle a is the third edge of the triangle. If we put $\gamma = \pi/2$ the construction is simplified. This will be demonstrated in Figure 4.

Remark: In Euclidean geometry there exist up to similarities exactly one right-angled isosceles triangle with $\alpha = \pi/2, \beta = \gamma = \pi/4$ and exactly one equilateral triangle with $\alpha = \beta = \gamma = \pi/3$. The construction in Figure 3 shows that in hyperbolic geometry every specification $\alpha = \pi/2, 0 < \beta = \gamma < \pi/4$ yields a right-angled isosceles triangle and every angle $\alpha < \pi/3$ yields an equilateral triangle.

5. A CONSTRUCTION OF A REGULAR *N*-GON WITH GIVEN ANGLE BETWEEN TWO ADJACENT EDGES IN THE POINCARÉ DISK

We construct a regular *n*-gon with given angle σ between adjacent edges in the Poincaré disk model. We assume that the center of the regular *n*-gon coincides with the center of the Poincaré disk. We draw a triangle (see section 4) with angles $\alpha = \pi / n$, $\beta = \sigma / 2$, $\gamma = \pi / 2$; *n*=5 and $\sigma = 60°$. The construction of *ABC* simplifies because the center of circles *a** and *a* coincide with *b* (Figure 4).

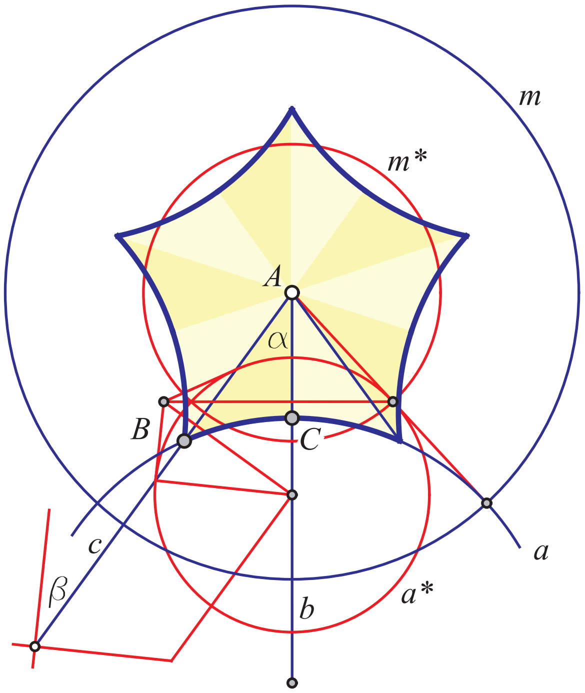

Figure 4: Right-angled triangle *ABC* with $\alpha = \pi / 5 = 36°$, $\beta = 30°$, $\gamma = \pi / 2$ and vertex *A* in the center and a regular pentagon in the Poincaré disk

The images from *ABC* under the reflexion across *b* and the rotations about *A* through $2k\pi / 5, k = 1,...,4$ build up the regular pentagon.

Remark: In Euclidean geometry only the regular hexagon is decomposable in six equilateral triangles. The situation in hyperbolic geometry is different. If a regular *n*-gon is decomposable in *n* equilateral triangles, then it is decomposable in 2*n* right-angled triangles with $\alpha = \pi / n$ and $\beta = 2\pi / n$ with $\alpha + \beta < \pi / 2$. From this it follows that for every $n > 6$ up to hyperbolic displacements a regular *n*-gon exists decomposable in *n* equilateral triangles. The construction is the same as in Figure 4 with specifications $\alpha = \pi / n, \beta = 2\pi / n, \gamma = \pi / 2$ with $n \geq 7$.

6. CONSTRUCTION OF A REGULAR *N*-GON WITH GIVEN SIDE LENGTH IN THE BELTRAMI-KLEIN MODEL

6.1 Equidistant curves in the Beltrami-Klein model

In the hyperbolic geometry an equidistant curve *c* is a circle with ultra-parallel diameters, all orthogonal to its axis *o*, while its center is an ultra-ideal point *O*. The equidistant curve intersects all its diameters in segments of the same length. In the Beltrami-Klein model axis *o* and center *O* are pole and polar with respect to the absolute circle *m*. If *o* is a diameter of *m*, then *O* is point at infinity in the underlying Euclidean plane.

6.2 Some right-angled triangles in the Beltrami – Klein model

We consider right-angled triangles in the hyperbolic geometry given by an acute angle α and length *a* of the opposite side and we assume that the vertex *A* is the center of *m* (Figure 5). Because an angle with vertex *A* in the center of the absolute circle has the same magnitude in Euclidean and in hyperbolic geometry we can use a Euclidean protractor to draw an angle α. Then one leg of α is the hypotenuse and the other one cathetus of the triangle on a diameter *o*. The second endpoint *B* of the hypotenuse has the distance *a* of *o*. Hence, *B* is a point of an equidistant curve *c* with axis *o*. If the hyperbolic length *a* is given by a segment then a hyperbolic reflexion exists that maps this segment on *AB'* on the diameter orthogonal to *o*. The point *B'* determines together with *o* as axis the equidistant curve c in the hyperbolic plane – an ellipse in Euclidean sense. The point of intersection of the curve c and the second leg of α is the second point B on the hypotenuse. Finally, the Euclidean right-angled triangle ABC is the requested hyperbolic right-angled triangle, too.

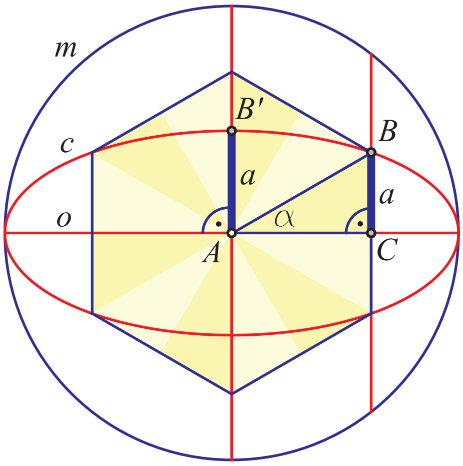

Figure 5 Right-angled triangle ABC with vertex A in the center of the absolute circle and a regular hexagon in the Beltrami-Klein model

7. CONSTRUCTION OF A REGULAR *N*-GON WITH GIVEN SIDE LENGTH IN THE BELTRAMI-KLEIN MODEL

Also in Figure 5 we construct a regular n-gon with given side length s in the Beltrami-Klein model. Its center coincides with the center of the Beltrami-Klein model. We specify $\alpha = \pi / n$, $s/2 = a$ and $n = 6$ and construct the right-angled triangle ABC. The images from ABC under the reflexion across o and the rotations about A through $2k\pi/6,\ k = 1,...,5$, form the regular hexagon.

8. CONCLUSIONS

Visualization of hyperbolic geometry in different models, i.e. the Poincaré disk model and the Beltrami-Klein model, makes the axiomatic development of hyperbolic geometry more lively. Computer programmes are available, but constructions by hand are a straight access to visualize geometry. As examples we constructed right-angled triangles and regular n-gons using elementary Euclidean constructions.

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LINKS

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