# Coupled Time-Domain Boundary and Finite Element Analysis with Non-conforming Interface Discretizations

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## Introduction

The Boundary Element Method (BEM) and the Finite Element Method (FEM) are rather complementary than competitive methods. The advantages of BEM are its applicability to large and semi-infinite domains with a low cost of discretization. Especially, the implicit fulfillment of the radiation condition make it very suitable for dynamic soil analyses. On the other hand its application to non-linear methods is very complicated, contrary to the FEM, which has proven to tackle most kinds of non-linearities appearing in mechanics. Therefore, it is desirable to make use of the advantages of both methods (cf. [7] for similar arguments). Domain Decomposition Methods provide such methodology by simply cutting the domain under consideration into sub-domains and using the appropriate discretizations scheme independently.
As shown below, in case of static problems both methods can be formulated as maps from the displacement (Dirichlet) boundary data to the traction (Neumann) boundary data, i.e., as *Dirichlet-to-Neumann* maps. By means of such maps, which only consider the boundaries of each sub-domain, variational principles for the problem can be given without specification of the method, which can be derived from boundary potentials [7]. Moreover, the interface conditions can be relaxed such that the requirement of coincident interface nodes is no longer necessary, as in the so-called Mortar Element Method [6].
In this work, the methodology for static problems will be briefly reviewed as presented by Steinbach [5] and then an extension to dynamic problems will be proposed.

## A Domain Decomposition Method for static problems

A static (or elliptic) mechanical boundary value problem is usually given in the form

$$\begin{aligned} (\mathcal{L}u)(x) &= f(x) \quad & x \in \Omega \\ u(y) &= \bar{u}(y) \quad & y \in \Gamma_D \\ q(y) = (\mathcal{T}u(x)) &= \bar{q}(y) \quad & y \in \Gamma_N \end{aligned} \tag{1}$$

with the differential operator $\mathcal{L}$ (the Lamé-Navier operator, for instance), the source term $f$, the given Dirichlet boundary data $\bar{u}$ and Neumann boundary data $\bar{q}$. $\Omega$ is the domain under consideration and its boundary is subdivided in the two disjoint parts $\Gamma_D$ and $\Gamma_N$. For simplicity, in the following only the homogeneous problem is considered, i.e., $f = 0$ (for finite element discretizations the consideration of body forces is easy to include).
Alternatively, one can write the boundary value problem (1) by means of the *Steklov-Poincaré* operator $\mathcal{S}$

$$(\mathcal{S}u)(y) = q(y) \quad y \in \Gamma \tag{2}$$

with the same boundary conditions as in (1). Refer to [5] for further details on the Steklov-Poincaré operator and on the rest of this section.
The discrete version of such an operator $\mathcal{S}$ is obtained by either a finite element or a boundary element discretization. In case of finite elements, it is the Schur complement (or static condensation) of the stiffness matrix

$$\begin{pmatrix} \mathsf{A}_{\mathrm{II}} & \mathsf{A}_{\mathrm{I}\Gamma} \\ \mathsf{A}_{\Gamma\mathrm{I}} & \mathsf{A}_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathsf{u}_{\mathrm{I}} \\ \mathsf{u}_{\Gamma} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathsf{f}_{\Gamma} \end{pmatrix} \quad \Rightarrow \quad \mathsf{S}\mathsf{u}_{\Gamma} = (\mathsf{A}_{\Gamma\Gamma} - \mathsf{A}_{\Gamma\mathrm{I}}\mathsf{A}_{\mathrm{II}}^{-1}\mathsf{A}_{\mathrm{I}\Gamma})\mathsf{u}_{\Gamma} = \mathsf{f}_{\Gamma}, \tag{3}$$

where the subscripts I and $\Gamma$ refer to location of the degrees of freedom in the interior or on the boundary of the domain, respectively.
In case of a boundary element discretization, consider the first integral equation

$$\mathsf{V}\mathsf{q} = \left(\frac{1}{2}\mathsf{I} + \mathsf{K}\right)\mathsf{u}_{\Gamma} \quad \Rightarrow \quad \mathsf{S}\mathsf{u}_{\Gamma} = \mathsf{B}\mathsf{V}^{-1}\left(\frac{1}{2}\mathsf{I} + \mathsf{K}\right)\mathsf{u}_{\Gamma} = \mathsf{B}\mathsf{q} = \mathsf{f}_{\Gamma} \tag{4}$$

with the discretized single layer potential $\mathsf{V}$, the discretized double layer potential $\mathsf{K}$, and a Galerkin weight matrix $\mathsf{B}$ which contains the inner products of the shape functions of the boundary displacements $u_\Gamma$ and of the tractions $q$. Note, that this discrete operator $\mathsf{S}$ is not symmetric, contrary to the finite element version from (3). But the use of symmetric Galerkin BEM with both integral equations would provide a symmetric $\mathsf{S}$.
Now, subdivision of the domain in, for instance, 2 sub-domains and discretization of each sub-domain independently with either finite or boundary elements leads to a variational principle of the form (the subscript $\Gamma$ of $u$ will be omitted)

$$\begin{aligned}
\int_{\Gamma_1} (S_1 u_1) v_1 \,\mathrm{d}s - \int_{\Gamma_{D1}} q_1 v_1 \,\mathrm{d}s - \int_{\Gamma_{12}} q_{12}(v_1 - v_2)\,\mathrm{d}s &= \int_{\Gamma_{N1}} \bar{q}_1 v_1 \,\mathrm{d}s \\
\int_{\Gamma_2} (S_2 u_2) v_2 \,\mathrm{d}s - \int_{\Gamma_{D1}} q_2 v_2 \,\mathrm{d}s + \int_{\Gamma_{12}} q_{12}(v_1 - v_2)\,\mathrm{d}s &= \int_{\Gamma_{N2}} \bar{q}_2 v_2 \,\mathrm{d}s \\
\int_{\Gamma_{D1}} u_1 p_1 \,\mathrm{d}s + \int_{\Gamma_{12}} (u_1 - u_2) p_{12} \,\mathrm{d}s &= \int_{\Gamma_{D1}} \bar{u}_1 p_1 \,\mathrm{d}s \\
\int_{\Gamma_{D2}} u_2 p_2 \,\mathrm{d}s - \int_{\Gamma_{12}} (u_1 - u_2) p_{12} \,\mathrm{d}s &= \int_{\Gamma_{D2}} \bar{u}_2 p_2 \,\mathrm{d}s
\end{aligned} \tag{5}$$

with $v_i$ and $p_i$ $(i = 1,2)$ the test functions corresponding to $u_i$ and $q_i$, respectively. Furthermore, the interface tractions $q_{12}$ are determined by sub-domain 1, i.e., $q_{12} = q_1 = -q_2$ for $y \in \Gamma_{12}$.

Note, that the interface tractions are mathematically the same as Lagrange multipliers. With this formulation all Dirichlet data are prescribed in a weak form, which holds for the boundary conditions and the interface displacements. Therefore, a strong continuity condition with the requirement of coincident interface nodes is omitted.
In the discretization of the variational principle (5), the Steklov-Poincaré will be replaced by either its finite or its boundary element version, as in expressions (3) and (4). Furthermore, it is not necessary to compute the discrete operator $\mathsf{S}$ directly but it suffices to replace it by the full system again and avoid the inversion of a matrix (cf. [5]).

## Extension to dynamics problems

In dynamical (or hyperbolic) problems, one has deal with initial boundary value problems of the form

$$\begin{aligned} \frac{\partial^2}{\partial t^2}u(x,t) - (\mathcal{L}u)(x,t) &= f(x,t) \quad x \in \Omega \quad t \in [0,\infty) \\ u(y,t) &= \bar{u}(y,t) \quad y \in \Gamma_D \quad t \in [0,\infty) \\ q(y,t) &= \bar{q}(y,t) \quad y \in \Gamma_N \quad t \in [0,\infty) \\ u(x,0) &= u_0(x) \quad x \in \Omega \\ \left(\frac{\partial}{\partial t}u\right)(x,0^+) &= v_0(x) \quad x \in \Omega\,, \end{aligned} \tag{6}$$

which is now equipped with the second time derivative of the unknown $u$ and initial conditions. A finite element discretization of the problem (6) can be obtained via the semi-discrete equation, as in, e.g., [3],

$$\mathsf{M}\ddot{\mathsf{u}}(t) + \mathsf{A}\mathsf{u}(t) = \mathsf{f}(t)\,,$$

which is solved with a time-integration scheme. $\mathsf{M}$ denotes the mass and $\mathsf{A}$ the stiffness matrix. This results in the discrete equation for $n$-th time step

$$\mathsf{A}^n\mathsf{u}^n = \mathsf{h}^n\,. \tag{7}$$

Here, the superscripts refer to the current time step and, choosing an equidistant time grid, the system matrix $\mathsf{A}^n$ can stay the same throughout computation. The right hand side $\mathsf{h}^n$ contains the force terms and the amount previous solutions relevant for the time integration scheme.
Similarly, in time-domain boundary elements the system of equations for the $n$-th time step is of the form (cf. [4])

$$\mathsf{V}^0\mathsf{q}^n + \sum_{i=1}^{n}\mathsf{V}^i\mathsf{q}^{n-i} = \left(\frac{1}{2}\mathsf{I} + \mathsf{K}^0\right)\mathsf{u}^n + \sum_{i=1}^{n}\mathsf{K}^i\mathsf{u}^{n-i}\,. \tag{8}$$

Both systems (7) and (8) can be manipulated in the same way as in the static case, refer to expressions (3) and (4), yielding a time-domain Dirichlet-to-Neumann map at the $n$-th time step

$$\mathsf{S}^n\mathsf{u}^n_\Gamma = \mathsf{f}^n\,. \tag{9}$$

Note, that $\mathsf{S}^n$ can stay the same for all time steps $n$, depending on the time integration and the material type.

## Outlook

The above presented methodology is a straightforward extension of non-conforming domain decomposition, which is fully established for static problems, to dynamic problems. Nevertheless, the computational effort is a big problem in this formulation and it is desirable to develop an efficient solution procedure for the occurring systems of equations. The newly developed $\mathcal{H}$-matrices seem promising, since they provide very cheap solution procedures, especially an $LU$-decomposition with quasi-linear complexity (cf. [1] and references therein). If the system matrices do not change throughout the dynamic analysis it would be very efficient to have a precomputed $LU$-decomposition at hand.

Another aspect is that the interface conditions employed in this formulation assume that the materials of the sub-domains be of similar type. Frequency domain acoustic-structure coupling has been presented in [2]. Time-domain fluid-structure coupling would definitely need a different treatment at this point.

## References


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