NONCONFORMING BETI/FETI METHOD FOR TIME DOMAIN ELASTODYNAMICS

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Summary A methodology for the combination of boundary and finite element discretizations for the numerical analysis of time-dependent problems is presented. The interface conditions arising from the partitioning of the problem are incorporated in a weak form by means of Lagrange multiplier fields and, therefore, allow for nonconform interface discretizations. The resulting system matrices have the same saddle point structure as in the FETI/BETI method.

PROBLEM STATEMENT

The linearized homogenous elastodynamic system of equations for an isotropic medium has the form [1]

$$\ddot{\mathbf{u}}(\mathbf{x},t) - c_s^2 \nabla(\nabla\cdot\mathbf{u}(\mathbf{x},t)) + c_p^2 \nabla\times(\nabla\times\mathbf{u}(\mathbf{x},t)) = \mathbf{0} \qquad (\mathbf{x},t)\in\Omega\times(0,\infty)\,, \tag{1}$$

where $\mathbf{u}$ is the displacement field, c_p and c_s denote the speeds of the pressure and shear wave, respectively, and Ω is the (possibly unbounded) domain under consideration. In addition initial conditions are prescribed at $t = 0$, which are here assumed to vanish, $\mathbf{u}(\mathbf{x},0) = \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{0}$, and boundary conditions on the Dirichlet and Neumann parts of the boundary Γ of the domain Ω are given

$$\mathbf{u}_\Gamma(\mathbf{y},t) = \mathbf{g}_D(\mathbf{y},t) \quad (\mathbf{y},t)\in\Gamma_D\times(0,\infty) \qquad\qquad \mathbf{t}(\mathbf{y},t) = \mathbf{g}_N(\mathbf{y},t) \quad (\mathbf{y},t)\in\Gamma_N\times(0,\infty)\,. \tag{2}$$

The boundary trace of $\mathbf{u}$ is abbreviated by $\mathbf{u}_\Gamma$ and the boundary tractions by $\mathbf{t}$.
Due to a geometrical subdivision by decomposition of the domain Ω into N_s subdomains, i.e., $\Omega = \bigcup_{r=1}^{N_s}\Omega^{(r)}$, the above stated problem is formulated locally on each subdomain $\Omega^{(r)}$ and equipped with the continuity condition

$$\mathbf{u}^{(r)}(\mathbf{y},t) - \mathbf{u}^{(p)}(\mathbf{y},t) = 0\,, \qquad (\mathbf{y},t)\in\Gamma^{(rp)}\times(0,\infty)\,, \tag{3}$$

where the subdomains $\Omega^{(r)}$ and $\Omega^{(p)}$ share the interface $\Gamma^{(rp)}$. In the static case of Eq. (1), i.e., the elastostatic system, the partitioned problem is represented by the variational saddle point formulation with a Lagrange multiplier field λ [2]

$$\begin{aligned}\int_{\Gamma^{(r)}} \left(\mathcal{S}^{(r)}\mathbf{u}_\Gamma^{(r)}\right)\cdot\mathbf{v}^{(r)}\mathrm{d}s - \sum_p \int_{\Gamma^{(rp)}} \lambda\cdot\left(\mathbf{v}^{(r)} - \mathbf{v}^{(p)}\right)\mathrm{d}s &= \int_{\Gamma_N^{(r)}} \mathbf{g}_N\cdot\mathbf{v}^{(r)}\mathrm{d}s \\ \sum_p \int_{\Gamma^{(rp)}} \mu\cdot\left(\mathbf{u}^{(r)} - \mathbf{u}^{(p)}\right)\mathrm{d}s &= 0\,.\end{aligned} \tag{4}$$

In this formulation, $\mathbf{v}^{(r)}$ and μ are the test functions corresponding $\mathbf{u}^{(r)}$ and λ. The operator $\mathcal{S}$ is the Steklov-Poincaré operator, which maps the Dirichlet datum $\mathbf{u}_\Gamma$ onto the Neumann datum $\mathbf{t}$ [2].

DISCRETIZATION METHODS

A classical finite element discretization of the local problems in elastodynamics yields the semi-discrete matrix equation

$$\mathsf{M}\ddot{\mathsf{u}}(t) + \mathsf{A}\mathsf{u}(t) = \mathsf{f}(t) \tag{5}$$

with the mass and stiffness matrices, M and A, the force vector f and the vectors of the time-dependent coefficients of the approximations for the displacement and acceleration fields, u and $\ddot{\mathsf{u}}$, see, e.g., [3]. Applying a time-stepping method (for instance the Newmark method) to Eq. (5), yields a system of equations, which can be abbreviated for the n-th time step

$$\begin{pmatrix}\tilde{\mathsf{A}}_{\mathrm{II}}^{(r)} & \tilde{\mathsf{A}}_{\mathrm{I}\Gamma}^{(r)} \\ \tilde{\mathsf{A}}_{\Gamma\mathrm{I}}^{(r)} & \tilde{\mathsf{A}}_{\Gamma\Gamma}^{(r)}\end{pmatrix}\begin{pmatrix}\mathsf{u}_{\mathrm{I},n}^{(r)} \\ \mathsf{u}_{\Gamma,n}^{(r)}\end{pmatrix} = \begin{pmatrix}\mathsf{f}_{\mathrm{I},n}^{(r)} \\ \mathsf{f}_{\Gamma,n}^{(r)}\end{pmatrix}\,, \tag{6}$$

where the degrees of freedom have been sorted according to their location in the interior of the subdomain $\Omega^{(r)}$ (subscript I) and on the interface (subscript Γ). The effect of the past time syeps are hidden in the right hand side. Now, the map from the boundary displacements to the boundary tractions as in Eq. (4) can be established by computation of the Schur complement of system (6)

$$\mathsf{S}_{fe}^{(r)}\mathsf{u}_{\Gamma,n}^{(r)} = \left(\tilde{\mathsf{A}}_{\Gamma\Gamma}^{(r)} - \tilde{\mathsf{A}}_{\Gamma\mathrm{I}}^{(r)}\left(\tilde{\mathsf{A}}_{\mathrm{II}}^{(r)}\right)^{-1}\tilde{\mathsf{A}}_{\mathrm{I}\Gamma}^{(r)}\right)\mathsf{u}_{\Gamma,n}^{(r)} = \mathsf{f}_{\Gamma,n}^{(r)} - \tilde{\mathsf{A}}_{\Gamma\mathrm{I}}^{(r)}\left(\tilde{\mathsf{A}}_{\mathrm{II}}^{(r)}\right)^{-1}\mathsf{f}_{\mathrm{I},n}^{(r)}\,. \tag{7}$$

By means of a time-domain boundary element discretization [4] of the r-th subdomain, the following system of equations is established, where only the first integral equation has been used

$$\begin{pmatrix} \mathsf{V}_0^{(r)} & -\left(\mathsf{C}^{(r)}+\mathsf{K}_0^{(r)}\right) \\ \mathsf{B}^{(r)} & \end{pmatrix}\begin{pmatrix} \mathsf{t}_n^{(r)} \\ \mathsf{u}_{\Gamma,n}^{(r)}\end{pmatrix} = \begin{pmatrix} \mathsf{f}_{D,n}^{(r)} \\ \mathsf{f}_{N,n}^{(r)}\end{pmatrix} - \sum_{k=1}^{n}\begin{pmatrix} \mathsf{V}_k^{(r)} & \mathsf{K}_k^{(r)}\end{pmatrix}\begin{pmatrix} \mathsf{t}_{n-k}^{(r)} \\ \mathsf{u}_{\Gamma,n-k}^{(r)}\end{pmatrix}. \tag{8}$$

The matrices V_k and K_k are the discretized single and double layer operators for time step k. C is the integral-free term and B a mass matrix. The vectors f_D and f_N refer to the prescribed Dirichlet and Neumann data, respectively. Again computing the Schur complement of system (8) gives a boundary element realization of the map from $\mathbf{u}_\Gamma$ to $\mathbf{t}$ at every time step.

Denoting these Schur complements by $\mathsf{S}_{fe}^{(r)}$ and $\mathsf{S}_{be}^{(p)}$ if a finite element discretization is applied to subdomain $\Omega^{(r)}$ and a boundary element discretization to subdomain $\Omega^{(p)}$, the following global system of equations is obtained for the example of $N_s = 2$, $r = 1$, and $p = 2$

$$\begin{pmatrix} \mathsf{S}_{fe}^{(1)} & & \mathsf{D}^{(1)} \\ & \mathsf{S}_{be}^{(2)} & \mathsf{D}^{(2)} \\ \left(\mathsf{D}^{(1)}\right)^\top & \left(\mathsf{D}^{(2)}\right)^\top & \end{pmatrix}\begin{pmatrix} \mathsf{u}_{\Gamma,n}^{(1)} \\ \mathsf{u}_{\Gamma,n}^{(2)} \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \mathsf{h}_{\Gamma,n}^{(1)} \\ \mathsf{h}_{\Gamma,n}^{(2)} \\ 0 \end{pmatrix}. \tag{9}$$

The matrices $\mathsf{D}^{(r)}$ are the interface matrices which are responsible for the incorporation of the interface condition (3) and $\mathsf{h}_{\Gamma,n}^{(r)}$ is the abbreviation for the effective boundary force terms in the n-th time step. System (9) has the same algebraic structure as in the FETI method [5]. Moreover, letting the approximation of the Lagrange multiplier field inherit its spatial discretization from one side of the interface only, the incorporation of nonconforming interface situations is straightforward [6].

NUMERICAL EXAMPLE

An example of the application of this methodology is the numerical analysis of an individual footing which is subject to a vertical step load on its upper surface and resting on an elastic halfspace. The discretization and the numerical outcome are shown in figure 1. Clearly, the discretization is non-conforming since neither the nodes are coincident nor

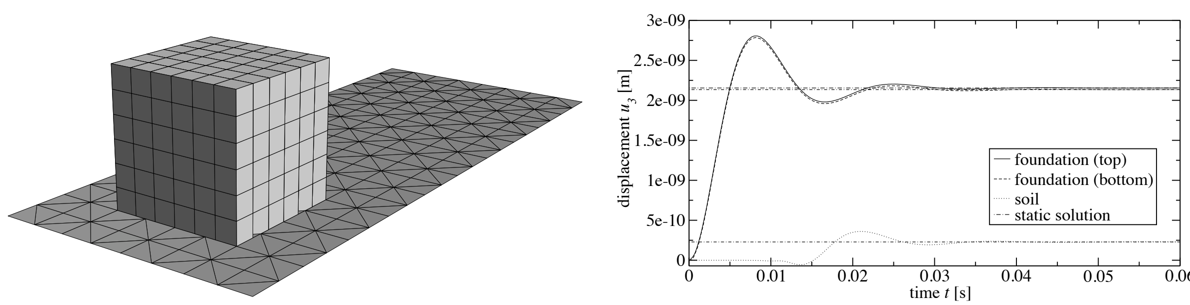

Figure 1. Numerical analysis of an individual footing under a dynamic step load — discretization by finite and boundary element methods (left) and vertical displacements against time for different positions compared with the corresponding static solution.

the approximation orders are equal for the different subdomains. The results show the vertical displacements at different points compared with the solution of the corresponding static problem. These points are the midpoint of the upper surface of the foundation (top), the midpoint of the lower surface (bottom), and a point on the surface of the soil at the far end of the discretization. The propagation of the disturbance along the surface is visible for the observation point on the soil surface, which consists of the pressure, shear and Rayleigh waves. Moreover, the results all converge to the static solution which indicates that the geometric damping of the halfspace is represented well and no spurious wave reflections occur.

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