

BOUNDARY ENTROPY SPECTRA AS FINITE SUBSUMS

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ABSTRACT. In this paper we provide a concrete construction of Furstenberg entropy values of τ -boundaries of the group $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_l}] \rtimes \{p_1^{n_1} \cdots p_l^{n_l} : n_i \in \mathbb{Z}\}$ by choosing an appropriate random walk τ . We show that the boundary entropy spectrum can be realized as the subsum-set for any given finite sequence of positive numbers.

CONTENTS

1.	Introduction	1
1.1.	Organization of the paper and ingredients of the proof of Theorem 1.1	2
1.2.	Acknowledgments	3
2.	Preliminaries	3
3.	Entropy observations	4
3.1.	Absolutely continuous measures provide equal information “on average”	4
3.2.	An entropy formula for certain measures	5
4.	An application	6
4.1.	A construction of absorbing measures to design entropy	8
4.2.	Proof of Theorem 1.1	10
	References	10

1. INTRODUCTION

A *random walk* on a countable infinite discrete group Γ is given by a probability measure τ on Γ , whose support generates the group as a semi-group. In 1963 Furstenberg introduced in [6] so-called *boundaries* for random walks. These are τ -stationary standard probability spaces (Y, η) on which Γ acts measurably such that for $\tau^{\otimes \mathbb{N}}$ -almost every sequence $(\gamma_i)_{i \in \mathbb{N}} \in \Gamma^{\mathbb{N}}$ the image measures $\gamma_1 \cdots \gamma_n \eta$ converge on a compact model to a Dirac measure. To quantify these spaces, Furstenberg introduced a notion of *entropy* defined as

$$h_\tau(Y, \eta) := \int_\Gamma \int_Y -\log \left(\frac{d\gamma^{-1}\eta}{d\eta}(y) \right) d\eta(y) d\tau(\gamma).$$

The boundary entropy spectrum and the entropy spectrum of (Γ, τ) are defined as

$$\text{BndEnt}(\Gamma, \tau) := \{h_\tau(Y, \eta) : (Y, \eta) \text{ a } \tau\text{-boundary}\}$$

and

$$\text{Ent}(\Gamma, \tau) := \{h_\tau(Y, \eta) : (Y, \eta) \text{ a } \tau\text{-stationary, ergodic probability } \Gamma\text{-space}\}.$$

The latter has been intensively studied, e.g. Lewis Bowen proved in [3] that the free group \mathbb{F}_r for $r \geq 2$ generators equipped with the uniform measure v has a *full entropy realization*, i.e. $\text{Ent}(\mathbb{F}_r, v) = [0, h(\mathbb{F}_r, v)]$ where $h(\Gamma, \tau) := \sum_{\gamma \in \Gamma} -\log(\tau(\gamma))\tau(\gamma)$ denotes the *random*

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walk entropy. The random walk entropy is always a natural upper bound for the entropy spectrum. Nevo showed in [10] that an infinite countable group with property (T) will always have a so-called *entropy gap* for any generating probability measure on the group, that is there is $\epsilon > 0$ (depending on the measure) such that any stationary ergodic space with entropy less than ϵ has entropy zero. Examples of groups without an entropy gap are for instance finitely generated virtually free groups with generating probability measures which have a finite first moment, shown by Hartman-Tamuz in [7]. The question about the shape of $\text{Ent}(\Gamma, \tau)$ is still widely open for many measured groups.

Here we shall investigate the shape of $\text{BndEnt}(\Gamma, \tau)$ for a concrete group Γ where we choose different measures τ to design various boundary entropy spectra. Note that

$$\text{BndEnt}(\Gamma, \tau) \subseteq \text{Ent}(\Gamma, \tau).$$

In our previous work [2], joint with Michael Björklund and Yair Hartman, we explicitly constructed random walks on a certain class of groups such that their boundary entropy spectra realize any given subsum sets. In particular, we constructed two measures τ_1 and τ_2 on the same group Γ such that $\text{BndEnt}(\Gamma, \tau_1) = [0, \text{Poi}(\Gamma, \tau_1)]$ while $\text{BndEnt}(\Gamma, \tau_2)$ is a Cantor set. The groups considered there were countable infinite direct sums of certain countable groups. In this short note we consider a group which is not a countable infinite direct sum and prove a certain subsum-realization for the boundary entropy spectrum using slightly different methods.

Given finitely many distinct primes, p_1, \dots, p_l , we study the group

$$\Gamma := \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_l} \right] \rtimes \{p_1^{n_1} \cdots p_l^{n_l} : n_i \in \mathbb{Z}\}.$$

Theorem 1.1. *Let β_1, \dots, β_l be positive real numbers. Then there exists a finitely supported generating probability measure τ on Γ such that*

$$\text{BndEnt}(\Gamma, \tau) = \left\{ \sum_{j \in J} \beta_j : J \subseteq \{1, \dots, l\} \right\}.$$

1.1. Organization of the paper and ingredients of the proof of Theorem 1.1

In Section 3 we express the entropy of probability measures which are absolutely continuous w.r.t. σ -finite product measures as sums of averaged *information functions* of the single measures of the product. Our proof uses a result by Anosov in [1]. Further in Corollary 4.5 we will see that the boundaries in concern meet the requirements for these entropy formulae. This is done by passing to a homogeneous setting of a certain locally compact, totally disconnected measured group by techniques we have developed in our previous work [2] with Michael Björklund and Yair Hartman. We will apply a theorem by Brofferio in [4], stating that the “maximal” boundary - the so-called *Poisson boundary* $\text{Poi}(\Gamma, \tau)$ - of a finitely generated subgroup Γ of the affine group $\mathbb{Q} \rtimes \mathbb{Q}^*$ equipped with a finitely supported, generating probability measure τ can be realized as

$$\text{Poi}(\Gamma, \tau) = \left(\prod_{\phi_p(\tau) < 0} \mathbb{Q}_p, \nu \right)$$

where ν is the uniquely τ -stationary probability measure on that space and $\phi_p(\tau)$ denotes the *p-drift* for τ (see Section 4 for a definition) running over all primes p including infinity (with $\mathbb{Q}_\infty := \mathbb{R}$). Note that the above product could be empty as well - e.g. if τ is symmetric - in which case we would see the one-point-space with the Dirac measure.

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2. PRELIMINARIES

Let G be a locally compact, second countable group and fix a left-Haar measure m_G on G . Throughout let μ be a probability measure on the Borel σ -algebra of G which is *admissible*, that is, it is *generating* in the sense that the semi-group generated by the support of μ is dense in G and it is *spread-out* meaning there exists some convolution power of μ which is absolutely continuous w.r.t. m_G . A standard probability spaces (X, ν) on which G acts jointly Borel measurably is called *G -space*. The action is *non-singular* if $g\nu$ and ν have the same null-sets for every $g \in G$. For these kind of actions Furstenberg [6] defined a notion of entropy

$$h_\mu(X, \nu) = \int_G \int_X -\log \left(\frac{dg^{-1}\nu}{d\nu}(x) \right) d\nu(x) d\mu(g)$$

provided that the *information function*

$$I_\nu : (g, x) \mapsto \log \left(\frac{dg^{-1}\nu}{d\nu}(x) \right)$$

lies in $L^1(G \times X, \mu \otimes \nu)$. For our purposes we may extend the definition of the information function to σ -finite measures on X .

A measure ν on a measurable G -space (X, \mathcal{B}) is called *μ -stationary* if $\nu = \mu * \nu$, i.e.

$$\nu(A) = \int_X \nu(g^{-1}A) d\mu(g)$$

for all $A \in \mathcal{B}$. Note that the assumption on (G, μ) to be admissible implies that every μ -stationary measure is in particular non-singular (see e.g. [11, Lemma 1.2]). Moreover μ -stationary probability measures give rise to measure-preserving transformations on the following product space.

Lemma 2.1 ([5, Proposition 1.3]). *Let ν be a μ -stationary probability measure on X . Then $T : G^{\mathbb{N}} \times X$ given by $T((g_n)_{n \geq 1}, x) := ((g_n)_{n \geq 2}, g_1 x)$ is a measurable map which preserves the measure $\mu^{\otimes \mathbb{N}} \otimes \nu$.*

The above introduced notion of entropy is of particular interest for the following spaces.

Definition 2.2. A compact probability space (X, ν) on which G acts continuously with ν being μ -stationary is called a *compact μ -boundary* if

$$g_1 \cdots g_n \nu \rightarrow \delta_{x((g_i)_{i \geq 1})} \quad \text{for } \mu^{\otimes \mathbb{N}}\text{-almost every } (g_i)_{i \geq 1} \in G^{\mathbb{N}}$$

in the weak*-topology, with $x((g_i)_{i \geq 1}) \in X$. A probability space which is measure-theoretical G -equivariantly isomorphic to a compact μ -boundary is called a *μ -boundary*. The *Poisson boundary* is the maximal such boundary in the sense that every other μ -boundary is a G -factor of it. (Its existence is provided by a result of Furstenberg, see e.g. [6]). We will write $\text{Poi}(G, \mu)$ to denote this space.

3. ENTROPY OBSERVATIONS

3.1. Absolutely continuous measures provide equal information “on average”

Let G be a locally compact, second countable group equipped with an admissible probability measure μ , and let (X, ν) be a G -space.

Proposition 3.1. *Let ν be a μ -stationary probability measure on X and let ξ be a non-singular σ -finite measure on X . If $\nu \ll \xi$ and if I_ν and I_ξ are both in $L^1(\mu \otimes \nu)$, then*

$$\int_G \int_X I_\nu(g, x) d\nu(x) d\mu(g) = \int_G \int_X I_\xi(g, x) d\nu(x) d\mu(g).$$

The proof of the above Proposition 3.1 relies on an old observation by Anosov [1] and will be provided at the end of this subsection.

Lemma 3.2 ([1, Theorem 1]). *Let (Z, \mathcal{Z}, ζ) be a probability space and $T : Z \rightarrow Z$ be a measure-preserving transformation on Z , i.e. $\zeta \circ T^{-1} = \zeta$. Given a measurable function $f : Z \rightarrow \mathbb{R}$ such that $f - f \circ T$ lies in $L^1(Z, \zeta)$, then*

$$\int_Z (f - f \circ T)(z) d\zeta(z) = 0.$$

Proof. Let us denote $h := f \circ T - f$. By Birkhoff’s Ergodic Theorem for ζ -a.e. $z \in Z$

$$\mathbb{E}[h|\mathcal{I}(T)](z) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} h(T^k(z)) = \lim_{n \rightarrow \infty} \frac{1}{n} (f(T^n(z)) - f(z)) = \lim_{n \rightarrow \infty} \frac{1}{n} f(T^n(z)),$$

where $\mathcal{I}(T)$ denotes the σ -algebra of T -invariant sets in \mathcal{Z} . Moreover, for every $\epsilon > 0$ we see that

$$\zeta\left(\left\{z \in Z : \left|\frac{f(T^n(z))}{n}\right| > \epsilon\right\}\right) = \zeta\left(\left\{z \in Z : \left|\frac{f(z)}{n}\right| > \epsilon\right\}\right) \rightarrow 0$$

by T -invariance of ζ . Hence (by using Borel-Cantelli) there exists a subsequence $(n_l)_{l \in \mathbb{N}}$ such that $\frac{f(T^{n_l}(z))}{n} \rightarrow 0$ for $l \rightarrow \infty$, for ζ -a.e. $z \in Z$. Thus

$$0 = \mathbb{E}[h|\mathcal{I}(T)]$$

and in particular

$$0 = \int_Z h d\zeta$$

since $\mathbb{E}[h|\mathcal{I}(T)] \in L^1(Z, \zeta)$. □

Corollary 3.3. *Let ν be a μ -stationary probability measure on X . For a measurable function $u : X \rightarrow \mathbb{R}$ such that $f(g, x) := u(x) - u(gx)$ belongs to $L^1(G \times X, \mu \otimes \nu)$ we have*

$$\int_G \int_X f(g, x) d\nu(x) d\mu(g) = 0.$$

Proof. Since ν is μ -stationary, by Lemma 2.1 we can consider the measure-preserving transformation $T : G^{\mathbb{N}} \times X$ defined as $T((g_n)_{n=1}^\infty, x) := ((g_n)_{n=2}^\infty, g_1 x)$. Let us set $\tilde{u} := u \circ \text{pr}_X$ where $\text{pr}_X : G^{\mathbb{N}} \times X \rightarrow X$ is the projection to X . Then clearly, for $g_1 = g$ we can write

$$u(x) - u(gx) = \tilde{u}(x) - \tilde{u}(T((g, g_2, g_3, \dots), x)).$$

Now to apply Lemma 3.2, it is left to show that

$$\tilde{u} - \tilde{u} \circ T \in L^1(G^{\mathbb{N}} \times X, \mu^{\otimes \mathbb{N}} \otimes \nu).$$

Indeed, \tilde{u} does only depend on the first coordinate of G and by assumption $u - u \circ g \in L^1(G \times X, \mu \otimes \nu)$, which ends the proof. □

Proof of Proposition 3.1. First note that $u := \frac{d\nu}{d\xi} > 0$, ν -almost everywhere. So

$$I_\nu(g, x) = \log \left(\frac{u(g(x)) dg^{-1} \xi}{u(x) d\xi}(x) \right) = -\log(u(x)) + \log(u(g(x))) + I_\xi(g, x)$$

for $\mu \otimes \nu$ -a.e. $(g, x) \in G \times X$. By Corollary 3.3 we see that

$$\int_{G \times X} \log(u(x)) - \log(u(g(x))) d\mu \otimes \nu(g, x) = 0$$

because $\log \circ u - \log \circ u \circ g = I_\nu - I_\xi \in L^1(G \times X, \mu \otimes \nu)$ by assumption. Thus we obtain

$$\int_{G \times X} I_\nu d\mu \otimes \nu = \int_{G \times X} I_\xi d\mu \otimes \nu.$$

□

3.2. An entropy formula for certain measures

In case when the μ -stationary probability measure ν is absolutely continuous w.r.t. to a product measure, we can rewrite the entropy as sums of averaged information functions of every single measure of the product.

Proposition 3.4. *Let $X = \prod_{i=1}^l X_i$ and G act diagonally on X such that ν is μ -stationary. Given non-singular σ -finite measures ξ_i on X_i such that $\nu \ll \bigotimes_{i=1}^l \xi_i$, then*

$$h_\mu(X, \nu) = \sum_{i=1}^l \int_G \int_X I_{\xi_i}(g, x_i) d\nu((x_n)_{n=1}^l) d\mu(g)$$

provided $I_\nu, I_{\xi_i} \in L^1(\mu \otimes \nu)$ for all $i = 1, \dots, l$.

In particular, if for every $g \in G$, for every $i \in \{1, \dots, l\}$ there exists a ξ_i -conull set $X'_i \subseteq X_i$ such that $\frac{dg^{-1} \xi_i}{d\xi_i}(x_i) = \Delta_i(g)$ for all $x_i \in X'_i$, then

$$h_\mu(X, \nu) = \sum_{i=1}^l \int_G -\log(\Delta_i(g)) d\mu(g).$$

The proof of the above proposition follows directly from Proposition 3.1 and the next basic fact about Radon-Nikodym derivatives on product spaces.

Lemma 3.5. *Let η_i and ζ_i be σ -finite measures on Y and Z , respectively, for $i = 1, 2$ such that $\eta_1 \ll \eta_2$ and $\zeta_1 \ll \zeta_2$. Then*

$$\frac{d(\eta_1 \otimes \zeta_1)}{d(\eta_2 \otimes \zeta_2)}(y, z) = \frac{d\eta_1}{d\eta_2}(y) \frac{d\zeta_1}{d\zeta_2}(z).$$

Proof of Proposition 3.4. By Proposition 3.1 we know that

$$h_\mu(X, \nu) = \int_G \int_X -\log \left(\frac{dg^{-1} \bigotimes_{i=1}^l \xi_i}{d \bigotimes_{i=1}^l \xi_i}(x) \right) d\nu(x) d\mu(g).$$

Since the action is diagonal we can apply Lemma 3.5 and obtain

$$h_\mu(X, \nu) = \sum_{i=1}^l \int_G \int_X -\log \left(\frac{dg^{-1} \xi_i}{d\xi_i}(x_i) \right) d\nu(x) d\mu(g).$$

□

4. AN APPLICATION

In this section we consider the group

$$\Gamma = \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_l} \right] \rtimes S$$

where $S := \{p_1^{n_1} \cdots p_l^{n_l} : n_i \in \mathbb{Z}\}$ for given distinct primes p_1, \dots, p_l .

As before we denote

$$\text{BndEnt}(\Gamma, \tau) := \{h_\tau(X, \nu) : (X, \nu) \text{ a } (\Gamma, \tau)\text{-boundary}\}.$$

Let us recall our main goal of this note, namely Theorem 1.1:

Theorem. *For any finite sequence of positive real numbers β_1, \dots, β_l there exist a finitely supported generating probability measure τ on Γ such that*

$$\text{BndEnt}(\Gamma, \tau) = \left\{ \sum_{j \in J} \beta_j : J \subseteq \{1, \dots, l\} \right\}.$$

Following [4] the p -drift of a measure τ on Γ is defined as

$$\phi_p(\tau) := \sum_{s \in S} \text{pr}_2 \tau(s) \log(|s|_p)$$

for a prime p . We say that τ has *negative drift* if its p -drift is negative for every $p \in \{p_1, \dots, p_l\}$.

Theorem 4.1 (in [4, Theorem 1]). *Let τ be a finitely supported, generating probability measure on $\Gamma = \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_l}] \rtimes \{p_1^{n_1} \cdots p_l^{n_l} : n_i \in \mathbb{Z}\}$ with negative drift. Then the Poisson boundary of (Γ, τ) is given by*

$$(\mathbb{Q}_{p_1} \times \dots \times \mathbb{Q}_{p_l}, \nu)$$

where ν is the unique τ -stationary probability measure on this space.

To prove Theorem 1.1 we will view $\mathbb{Q}_{p_1} \times \dots \times \mathbb{Q}_{p_l}$ as a homogeneous space w.r.t. a completion of Γ . We can diagonally embed

$$\rho : \Gamma \hookrightarrow G := \mathbb{Q}_{p_1} \times \dots \times \mathbb{Q}_{p_l} \rtimes S$$

where S denotes $\{p_1^{n_1} \cdots p_l^{n_l} : n_i \in \mathbb{Z}\}$. Note that the above embedding has a dense image, since $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_l}]$ is dense in \mathbb{Q}_{p_i} for every $p_i \in \{p_1, \dots, p_l\}$. Indeed, by the Theorem of Strong Approximation (e.g. [12, Theorem 5.8]), for every $a_i \in \mathbb{Q}_{p_i}$, $\forall \epsilon > 0$ there exists $x \in \mathbb{Q}$ such that $\|x - a_i\|_{p_i} < \epsilon \forall i = 1, \dots, l$ and $\|x\|_p \leq 1$ for all primes p except $\{p_1, \dots, p_l\}$. The latter property implies that $x \in \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_l}]$.

Let us set

$$X := \mathbb{Q}_{p_1} \times \dots \times \mathbb{Q}_{p_l}.$$

Then, naturally, $G \curvearrowright X$ continuously and $X \cong G/\hat{S}$ for $\hat{S} := \{0\} \rtimes S$ becomes a homogeneous space w.r.t. the action of G .

Lemma 4.2. *Let ν be a non-singular probability measure on X . Then every G -factor of (X, ν) is of the form*

$$\left(\prod_{j \in J} \mathbb{Q}_{p_j}, \pi_J \nu \right)$$

for some index subset $J \subseteq \{1, \dots, l\}$, where $\pi_J : X \rightarrow \prod_{j \in J} \mathbb{Q}_{p_j}$ denotes the given factor map.

Proof. Let $(Y, \pi\nu)$ be a measurable G -factor of (X, ν) with surjective G -equivariant factor map π . Since $X = G/(\{0\} \times S)$, there exists a closed subgroup C of G which contains $\{0\} \times S$ such that $(Y, \pi\nu) \cong (G/C, \phi\nu)$ as G -spaces (mod 0), where $\phi : G/(\{0\} \times S) \rightarrow G/C$ denotes the canonical factor map, confer e.g. [9, Chapter 4, Section 2, Proposition 2.4 (b)]. Now, if C is properly larger than $\{0\} \times S$, then there exists $(r, s) \in C$ with $r \neq (0, \dots, 0)$ in X , i.e. there is some $j \in \{0, \dots, l\}$ such that the j -th coordinate r_j of r is non-zero in \mathbb{Q}_{p_j} , then since $(\tilde{s}r, 1) = (0, \tilde{s})(r, s)(0, s^{-1}\tilde{s}^{-1}) \in C$ for any $\tilde{s} \in S$ we obtain that the j -th coordinate in C contains all elements of the group generated by $\{\tilde{s}r_j : \tilde{s} \in S\}$ which is exactly $r_j\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_l}]$. Since C is closed and $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_l}]$ is dense in \mathbb{Q}_{p_j} we obtain that the j -th coordinate of C has to be whole \mathbb{Q}_{p_j} . The same argument goes through for any non-zero coordinate entry of C such that in the end we obtain $Y \cong G/(\prod_{j \in J^c} \mathbb{Q}_{p_j}) \cong \prod_{j \in J} \mathbb{Q}_{p_j}$ as G -spaces for some index set $J \subseteq \{1, \dots, l\}$, such that the measure $\pi\nu$ on Y is transformed to $\text{pr}_J \nu$ on $\prod_{j \in J} \mathbb{Q}_{p_j}$. \square

In order to relate now the boundary theory of (Γ, τ) with those of (G, μ) we need to require some relations for the measures in charge. For this end let us consider the following subgroups

$$\Lambda := \mathbb{Z} \rtimes \{1\} \leq \Gamma \text{ and } K := \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_l} \rtimes \{1\}.$$

We remark that (Γ, Λ) is a so-called reduced Hecke pair - i.e. has finite Λ -orbits on Γ/Λ - and (G, K) its Schlichting completion, see e.g. [8] for details and definitions. Important to notice is that the coset spaces Γ/Λ and G/K are Γ -equivariant isomorphic. Such an isomorphism is for instance given by $\theta : \Gamma/\Lambda \rightarrow G/K$ with $\gamma\Lambda \mapsto \rho(\gamma)K$, see [8, Proposition 3.9].

Given a probability measure τ on Γ we want to construct a probability measure μ_τ on G such that they coincidence on the coset spaces Γ/Λ and G/K , in a sense that

$$\tau(\gamma\Lambda) = \mu_\tau(\vartheta(\gamma\Lambda)), \quad \forall \gamma\Lambda \in \Gamma/\Lambda. \quad (4.1)$$

Given $\tau \in \text{Prob}(\Gamma)$ we set

$$\mu_\tau(f) := \sum_{\gamma\Lambda \in \Gamma/\Lambda} \tau(\gamma\Lambda) \int_K f(\rho(\gamma)k) dm_K(k) \quad (4.2)$$

for $f : G \rightarrow \mathbb{R}$ measurable, where m_K denotes the Haar measure on K with $m_K(K) = 1$. The above is well-defined due to left- $\rho(\Lambda)$ -invariance of m_K . Remark that equation 4.1 is indeed fulfilled in this construction.

Measures which are Λ -invariant on the coset spaces Γ/Λ provide a theory of relating (G, μ_τ) -stationary spaces to (Γ, τ) -stationary ones. As in our previous work [2] we may give these measures a name.

Definition 4.3. We call a probability measure τ on Γ Λ -*absorbing* if

$$\tau(\lambda\gamma\Lambda) = \tau(\gamma\Lambda), \quad \forall \lambda \in \Lambda$$

for all $\gamma \in \Gamma$.

Using these measures in Brofferio's result in Theorem 4.1 we obtain now a homogeneous setting. Let us fix a Haar measure $m_{\mathbb{Q}_{p_i}}$ on \mathbb{Q}_{p_i} and let μ_τ denote the probability measure on G in equation 4.2 corresponding to τ .

Corollary 4.4. *Let τ be a finitely supported, generating probability measure on Γ which is Λ -absorbing and has negative drift. Then*

$$\text{Poi}(G, \mu_\tau) = \text{Poi}(\Gamma, \tau) = \left(\prod_{i=1}^l \mathbb{Q}_{p_i}, \nu \right)$$

where ν is the unique τ -stationary and unique μ_τ -stationary probability measure on $\prod_{i=1}^l \mathbb{Q}_{p_i}$. Moreover $\nu \ll \bigotimes_{i=1}^l m_{\mathbb{Q}_{p_i}}$.

Proof. Let ν be the unique τ -stationary probability measure given by Theorem 4.1. It is not hard to show that every μ_τ -stationary probability measure is τ -stationary (see [2, Theorem 1.6 (P3)]). Thus there can be at most one μ_τ -stationary probability measure on $\mathbb{Q}_{p_1} \times \dots \times \mathbb{Q}_{p_l}$ and it has to coincide with ν . The existence of a μ_τ -stationary probability measures can be shown along the same lines as Brofferio's construction in [4] for τ -stationary probability measures. Note that μ_τ is generating for G since τ is generating Γ ([2, Theorem 1.6 (P2)]). In particular ν is G -quasi-invariant. Since on homogeneous spaces there is a unique G -quasi-invariant measure class (see e.g. [13, Theorem 5.19]), we obtain that $\nu \ll \bigotimes_{i=1}^l m_{\mathbb{Q}_{p_i}}$. \square

In [2, Corollary 4.15], it is shown that every Γ -factor of a G -ergodic space is (Γ -equivariant) isomorphic to a G -factor, thus from Lemma 4.2 and Corollary 4.4 we can conclude

Corollary 4.5. *Let τ be a Λ -absorbing, finitely supported, generating probability measure on Γ with negative drift. Then all τ -boundaries of Γ are of the form*

$$\left(\prod_{j \in J} \mathbb{Q}_{p_j}, \eta_J \right)$$

for some index set $J \subseteq \{1, \dots, l\}$ where Γ acts on $\prod_{j \in J} \mathbb{Q}_{p_j}$ via the embedding in G and $\eta_J \ll \bigotimes_{j \in J} m_{\mathbb{Q}_{p_j}}$.

The above Corollary 4.4 together with the entropy observation in Proposition 3.4 will yield to a proof of Theorem 1.1, provided we construct τ such that it is finitely supported, Λ -absorbing and has negative drift. Let us do this construction for more general semi-directs products in the next subsection.

4.1. A construction of absorbing measures to design entropy

Here R shall be a countable ring with 1 and S a multiplicative subgroup of R . Let us consider a Hecke pair (Γ, Λ) of the form

$$\Gamma = R \rtimes S \text{ and } \Lambda = U \rtimes \{e\}$$

where U is an additive subgroup of R . Note that (Γ, Λ) is a Hecke pair iff $[U : sU \cap U] < \infty$ for every $s \in S$.

We will construct a finitely supported, generating and Λ -absorbing probability measure τ on Γ that is Λ -invariant and of the following form

$$\tau(r, s) = \kappa_s(r) \iota(s) \tag{4.3}$$

for probability measures κ_s on R and ι on S . In order to obtain that τ is Λ -absorbing, i.e.

$$\tau((u, e)(r, s)\Lambda) = \tau((r, s)\Lambda), \quad \forall u \in U$$

we need

$$\kappa_s(u + r + sU) = \kappa_s(r + sU), \quad \forall u \in U \tag{4.4}$$

for every $r \in R$ and $s \in S$ (with $\iota(s) \neq 0$). To this end let $V_s \subseteq R$ be a set of representatives for $U/(sU \cap U)$ with $0 \in V_s$, where the cosets are taken w.r.t. addition. By the Hecke assumption V_s is finite for every $s \in S$. Similar as in [2, Remark 1.15], we may thus construct Λ -absorbing probability measure in the following way.

Lemma 4.6. *Let κ be a probability measure on R . Then*

$$\kappa_s(t) := \frac{1}{|V_s|} \sum_{v_s \in V_s} \kappa(t + v_s), \quad t \in R$$

is a probability measure on R fulfilling equation 4.4 for every $s \in S$. In particular, if Γ is finitely generated, then there exists a finitely supported, generating, Λ -absorbing probability measure τ on Γ of the form 4.3.

Proof. We need to show that κ_s is U -invariant on every coset $r + sU$, hence that

$$\sum_{v_s \in V_s} \kappa(u_0 + r + sU + v_s) = \sum_{v_s \in V_s} \kappa(r + sU + v_s) \text{ for every } u_0 \in U.$$

Let W_s be a set of representatives of $sU/(sU \cap U)$. In particular

$$sU = \bigsqcup_{w_s \in W_s} (w_s + sU \cap U) \text{ and } U = \bigsqcup_{v_s \in V_s} (v_s + sU \cap U),$$

hence

$$\bigsqcup_{v_s \in V_s} (sU + v_s) = \bigsqcup_{v_s \in V_s} \bigsqcup_{w_s \in W_s} (w_s + sU \cap U + v_s) = \bigsqcup_{w_s \in W_s} (w_s + U).$$

This implies $\sum_{v_s \in V_s} \kappa(u_0 + r + sU + v_s) = \sum_{w_s \in W_s} \kappa(r + U + w_s) = \sum_{v_s \in V_s} \kappa(r + sU + v_s)$, which ends the proof of the first statement.

It is left to show that we can choose ι and κ such that τ given by equation 4.3 is finitely supported and generating, whenever Γ is finitely generated. Let E be a finite set of generators of Γ . We shall find finitely supported probability measures ι on S and κ on R such that

$$\text{supp}(\tau) = \{(r, s) : r \in \text{supp}(\kappa_s), s \in \text{supp}(\iota)\} \supseteq E. \quad (4.5)$$

First, we choose a finitely supported probability measure ι on S such that

$$\text{supp}(\iota) \supseteq \text{pr}_S(E). \quad (4.6)$$

Such a measure always exists, for instance the uniform measure on $\text{pr}_S(E)$. Further, for every $s \in S$ we see that $\text{supp}(\kappa_s) \supseteq \text{supp}(\kappa)$ because $0 \in V_s$ by assumption. Thus to obtain 4.5 it suffices to take κ to be a finitely supported probability measure with support containing $\text{pr}_R(E)$, for example the uniform probability measure on $\text{pr}_R(E)$. \square

Now let us go back to the case

$$R = \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_l}] \text{ and } S = \{p_1^{n_1} \cdots p_l^{n_l} : n_i \in \mathbb{Z}\}.$$

Lemma 4.7. *Given $\beta_1, \dots, \beta_l > 0$ we can find a finitely supported, generating, Λ -absorbing probability measure τ on Γ such that*

$$\phi_{p_i}(\tau) = -\beta_i$$

for any $i = 1, \dots, l$.

Proof. We will construct τ as in equation 4.3. Recall that $(S, \cdot) \cong (\mathbb{Z}^l, +)$. We may thus set ι to be a product measure

$$\iota = \bigotimes_{i=1}^l \iota_i$$

with ι_i being probability measures on \mathbb{Z} . With τ as in equation 4.3 we thus obtain

$$\phi_{p_i}(\tau) = \sum_{n \in \mathbb{N}} \iota_i(n) n \log(p_i).$$

Hence we shall find finitely supported, generating probability measures ι_i such that

$$\sum_{n \in \mathbb{Z}} \iota_i(n) n \log(p_i) = \frac{\beta_i}{\log(p_i)} =: \alpha. \quad (4.7)$$

Let σ be a finitely supported, generating probability measure on \mathbb{Z} with non-zero mean $\mathbb{E}[\sigma] = \sum_{n \in \mathbb{Z}} \sigma(n)n$, for example $\sigma = \frac{3}{4}\delta_1 + \frac{1}{4}\delta_{-1}$. For every i choose N_i such that $\frac{\beta_i}{\mathbb{E}[\sigma]} < N_i$. By setting now

$$\nu_i := \frac{\beta_i}{\mathbb{E}[\sigma]N_i} \sigma^{*N_i} + \left(1 - \frac{\beta_i}{\mathbb{E}[\sigma]N_i}\right) \delta_0$$

we obtain an instance of a generating, finitely supported probability measure fulfilling equation 4.7 since $\mathbb{E}[\sigma^{*N_i}] = N_i \mathbb{E}[\sigma]$.

Recall that in order to obtain $\tau(r, s) := \kappa_s(r)\nu(s)$ to be Λ -absorbing, finitely supported and generating, we have seen in the proof of Lemma 4.6, that the only constraint on ν is that $\text{supp}(\nu) \subseteq \text{pr}_S(E)$ (equation 4.6) for a finite generating set E of Γ . To guarantee this we can just choose the above numbers N_i so large that $\prod_{i=1}^l \text{supp}(\sigma^{*N_i}) \supseteq \text{pr}_S(E)$ which gives $\text{supp}(\nu) \supseteq \text{pr}_S(E)$ since $\text{supp}(\nu_i) = \text{supp}(\sigma^{*N_i}) \cup \{0\}$. \square

4.2. Proof of Theorem 1.1

Let Γ, G, Λ as in Theorem 1.1 and set $X := \mathbb{Q}_{p_1} \times \dots \times \mathbb{Q}_{p_l}$.

Proof of Theorem 1.1. We start with the ansatz

$$\tau(r, s) := \kappa_s(r) \bigotimes_{i=1}^l \nu_i(s)$$

where we identify $(S, \cdot) \cong (\mathbb{Z}^l, +)$ to let $\nu = \bigotimes_{i=1}^l \nu_i$ be a product measure on S with probability measures ν_i on \mathbb{Z} . By Lemma 4.6 we can choose probability measures ν_i on \mathbb{Z} and κ_s on $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_l}]$, such that τ becomes a Λ -absorbing, finitely supported and generating probability measures with the property that

$$\phi_{p_i}(\tau) = -\beta_i, \quad \forall i. \quad (4.8)$$

In particular the drift is negative and thus by Corollary 4.5 we know that every τ -boundary is of the form

$$(Y_J, \eta_J) := \left(\prod_{j \in J} \mathbb{Q}_{p_j}, \pi_J \nu \right)$$

for some index set $J \subseteq \{1, \dots, l\}$ for ν given by Theorem 4.1 with $\pi_J \nu \ll \bigotimes_{j \in J} m_{\mathbb{Q}_{p_j}}$. Thus by Proposition 3.4 we obtain

$$h_\tau(Y_J, \eta_J) = \sum_{j \in J} \int_{\Gamma} -\log(|s|_{p_j}) d\tau(r, s) = \sum_{j \in J} -\phi_{p_j}$$

since

$$\frac{d(r, s)^{-1} m_{\mathbb{Q}_{p_j}}}{dm_{\mathbb{Q}_{p_j}}} = |s|_{p_j}.$$

Thus by equation 4.8 we obtain $h_\tau(Y_J, \eta_J) = \sum_{j \in J} \beta_j$. \square

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