

RANDOM WALKS ON DENSE SUBGROUPS OF LOCALLY COMPACT GROUPS

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ABSTRACT. Let Γ be a countable discrete group, H a lcsc totally disconnected group and $\rho : \Gamma \rightarrow H$ a homomorphism with dense image. We develop a general and explicit technique which provides, for every compact open subgroup $L < H$ and bi- L -invariant probability measure θ on H , a Furstenberg discretization τ of θ such that the Poisson boundary of (H, θ) is a τ -boundary. Among other things, this technique allows us to construct examples of finitely supported random walks on certain lamplighter groups and solvable Baumslag-Solitar groups, whose Poisson boundaries are prime, but not L^p -irreducible for any $p \geq 1$, answering a conjecture of Bader-Muchnik in the negative. Furthermore, we give an example of a countable discrete group Γ and two spread-out probability measures τ_1 and τ_2 on Γ such that the boundary entropy spectrum of (Γ, τ_1) is an interval, while the boundary entropy spectrum of (Γ, τ_2) is a Cantor set.

1. INTRODUCTION

1.1. Furstenberg discretizations

Let μ be a Borel probability measure on a locally compact and second countable (lcsc) group G . We say that μ is *spread-out* if it is absolutely continuous with respect to the Haar measure on G and if its support generates G as a semigroup. If μ is spread-out, we say that μ is a *random walk* on G , and we refer to the pair (G, μ) as a *measured group*. Let (X, \mathcal{B}_X) be a measurable space, endowed with a jointly Borel measurable action of G , and let ξ be a probability measure on \mathcal{B}_X . We say that ξ is μ -*stationary* if $\mu * \xi = \xi$. If ξ is μ -stationary, (X, ξ) is a *Borel (G, μ) -space*.

Let Γ be a countable discrete group and let H be a lcsc group. Suppose that $\rho : \Gamma \rightarrow H$ is a homomorphism with dense image. Suppose that θ is a spread-out Borel probability measure on H . We say that a spread-out probability measure τ on Γ is a *Furstenberg discretization* of θ (with respect to ρ) if every Borel (H, θ) -space is also a (Γ, τ) -space, where Γ acts via ρ (equivalently, τ is a Furstenberg discretization of θ if the Poisson boundary of (H, θ) is τ -stationary). The following theorem is due to Furstenberg [32], see also [50, Chapter VI, Proposition 4.1] for a more detailed exposition.

Theorem 1.1 (Furstenberg). *Suppose that H is compactly generated and θ is compactly supported. Then θ admits a Furstenberg discretization τ with respect to ρ .*

Remark 1.2. In [32], Furstenberg only considered the setting when $\rho(\Gamma)$ is a *lattice subgroup* in H , so in particular, $\rho(\Gamma)$ is not dense in H . Hence our version of Theorem 1.1 is not explicitly stated in [32] or [50], but it can be readily proved along the same lines as [50, Chapter VI, Proposition 4.1], using that the Haar measurable function $L : G \rightarrow [0, \infty)$ constructed there is left-invariant under $\rho(\Gamma)$, whence almost surely constant since $\rho(\Gamma)$ is dense in G .

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In the case when $\rho(\Gamma)$ is a lattice, Furstenberg instead employs an ingenious convexity argument to ensure that L is almost surely constant. Once it has been established that L is constant, the rest of the argument goes through as before.

Before we move on to the main theme of this paper, let us briefly list some problematic aspects concerning the construction of Furstenberg discretizations in Theorem 1.1.

- (A1) In general we cannot guarantee that τ is finitely supported (if Γ is a finitely generated group).
- (A2) It is not known whether τ can always be chosen so that the Poisson boundary of (H, θ) is τ -proximal (cf. Remark 5.1).
- (A3) Since the construction of τ from θ is very indirect, explicit computations with τ are usually very demanding. For instance, it seems like a daunting task in general to compute the Furstenberg entropy of a Borel (H, θ) -space with respect to τ .

Remark 1.3. In the setting when $\rho(\Gamma)$ is a lattice, the aspect (A2) has been addressed (and affirmatively answered) in many special cases, see e.g. [8, 23, 24, 27, 48]. In fact, in these cases, the Poisson boundary of (H, θ) is also the Poisson boundary of (Γ, τ) . We stress that this may not longer be the case if $\rho(\Gamma)$ is dense in H . For instance, if $\Gamma = \mathrm{SL}_2(\mathbb{Q})$, $H = \mathrm{SL}_2(\mathbb{R})$ and ρ is the standard inclusion, then for every spread-out probability measure θ on H and for *every* Furstenberg discretization τ of θ , the Poisson boundary of (H, θ) is *never* a maximal τ -boundary (this is because the embeddings of Γ into $\mathrm{SL}_2(\mathbb{Q}_p)$ for different primes p always contribute to the Poisson boundary of Γ ; see e.g. [18] for a more detailed discussion about this point).

In this paper we study Furstenberg discretizations in the case when H is *totally disconnected*. Given a dense embedding ρ of a countable group Γ into a totally disconnected lsc group H , and a compact and open subgroup L of H , we shall introduce a convex set of probability measures on Γ (called absorbing measures) and define an explicit *surjective* affine map from the set of absorbing measures *onto* the set of bi- L -invariant probability measures on H such that every absorbing measure τ on Γ is a Furstenberg discretization (with respect to ρ) of the image measure θ_τ under this affine map.

Our novel construction is summarized in Theorem 1.6 and Corollary 1.9 below, and some sample applications are given in Theorem 1.13 and Theorem 1.19.

1.2. Hecke pairs and absorbing measures

We shall now introduce the key players in this paper: *absorbing measures*. Let Γ be a countable group, and let Λ be a subgroup of Γ . We denote by $\mathrm{Prob}(\Gamma)$ and $\mathrm{Prob}(\Gamma/\Lambda)$ the space of probability measures on Γ and Γ/Λ respectively, and we write $\mathrm{Prob}(\Gamma/\Lambda)^\Lambda$ for the subset of $\mathrm{Prob}(\Gamma/\Lambda)$ consisting of Λ -invariant probability measures. Let $\alpha : \Gamma \rightarrow \Gamma/\Lambda$ be the canonical projection map, and write α_* for the induced map between $\mathrm{Prob}(\Gamma)$ and $\mathrm{Prob}(\Gamma/\Lambda)$, given by

$$\bar{\tau}(\gamma\Lambda) := \alpha_*\tau(\gamma\Lambda) = \sum_{\lambda \in \Lambda} \tau(\gamma\lambda), \quad \text{for } \gamma\Lambda \in \Gamma/\Lambda.$$

The set $\mathrm{Prob}(\Gamma; \Lambda)$ of Λ -*absorbing probability measures* on Γ is defined by

$$\mathrm{Prob}(\Gamma; \Lambda) = \{\tau \in \mathrm{Prob}(\Gamma) : \alpha_*\tau \in \mathrm{Prob}(\Gamma/\Lambda)^\Lambda\}.$$

Note that if Λ is *normal*, then every probability measure on Γ is Λ -absorbing.

We say that (Γ, Λ) is a *Hecke pair* if for every $\gamma \in \Gamma$, the subgroup $\Lambda \cap \gamma \Lambda \gamma^{-1}$ has finite index in Λ ; or equivalently, if every Λ -orbit in Γ/Λ is finite. In this case, it is also common to refer to Λ as an *almost normal* (or *commensurated*) subgroup of Γ .

We collect some basic properties of $\text{Prob}(\Gamma; \Lambda)$ in the following theorem.

Theorem 1.4. *Let Γ be a countable group and let Λ be a subgroup of Γ .*

(I) *For all $\tau_1, \tau_2 \in \text{Prob}(\Gamma; \Lambda)$,*

$$\alpha_*(\tau_1 * \tau_2)(\gamma\Lambda) = \sum_{\eta\Lambda} \bar{\tau}_1(\eta\Lambda) \bar{\tau}_2(\eta^{-1}\gamma\Lambda), \quad \text{for every } \gamma\Lambda \in \Gamma/\Lambda.$$

*In particular, $(\text{Prob}(\Gamma; \Lambda), *)$ is a monoid, where δ_e is the identity element.*

(II) *If Γ acts by linear isometries on a Banach space E and the dual action preserves a weak*-closed convex subset C of E^* , then*

$$\text{Prob}(\Gamma; \Lambda) * C^\Lambda \subseteq C^\Lambda,$$

where C^Λ denotes the (possibly empty) set of Λ -invariant points in C . In particular, if C is weak-compact and Λ is amenable, then for every $\tau \in \text{Prob}(\Gamma; \Lambda)$, there exists $c \in C^\Lambda$ such that $\tau * c = c$.*

(III) *If (Γ, Λ) is a Hecke pair, then for every subset $S \subset \Gamma$, there exist*

(i) a subset $\tilde{S} \subset \Gamma$,

(ii) an affine map $\text{Prob}(S) \rightarrow \text{Prob}(\Gamma; \Lambda) \cap \text{Prob}(\tilde{S})$, $\tau \mapsto \tilde{\tau}$,

such that $\text{supp}(\tau) \subseteq \text{supp}(\tilde{\tau})$. If S is finite, then \tilde{S} can be chosen finite. In particular, if τ is spread-out, then so is $\tilde{\tau}$.

The main point of (III) is to show that if Γ is finitely generated and (Γ, Λ) is a Hecke pair, then finitely supported, spread-out and Λ -absorbing measures exist in abundance.

1.3. Hecke measured groups and their completions

Hecke pairs can be constructed along the following lines. Given

- a countable discrete group Γ ,
- a lscs *totally disconnected group* H and a compact open subgroup $L < H$,
- a homomorphism $\rho : \Gamma \rightarrow H$ with dense image,

we set $\Lambda := \rho^{-1}(L) < \Gamma$, and note that

$$\Lambda \cap \gamma \Lambda \gamma^{-1} = \rho^{-1}(L \cap \rho(\gamma)L\rho(\gamma)^{-1}), \quad \text{for all } \gamma \in \Gamma.$$

Since L is compact and open, the intersection $L \cap \rho(\gamma)L\rho(\gamma)^{-1}$ is a compact and open subgroup of L , and has thus finite index in L for every $\gamma \in \Gamma$, whence (Γ, Λ) is a Hecke pair. We say that (H, L, ρ) is a *completion triple* of the Hecke pair (Γ, Λ) . Conversely, Tzanev [60] has proved (based on some ideas of Schlichting [58]) that if (Γ, Λ) is a Hecke pair, then there is always a completion triple (H, L, ρ) of (Γ, Λ) . This completion triple is often referred to as the *Schlichting (or relatively profinite) completion* of (Γ, Λ) in the literature, and we shall discuss several examples below.

The key point of our first main result, and its corollaries, is that if (Γ, Λ) is a Hecke pair and (H, L, ρ) is a completion triple of (Γ, Λ) , then there is a natural affine map between spread-out and Λ -absorbing probability measures on Γ and spread-out and bi- L -invariant probability measures on H , which

- respects convolutions,
- for every Λ -absorbing spread-out probability measure τ on Γ produces a spread-out bi- L -invariant probability measure θ_τ on H such that τ is a Furstenberg discretization of θ_τ .
- induces a Γ -equivariant and measure-preserving map between the Poisson boundary of (Γ, τ) and the Poisson boundary of (H, θ_τ) . In some cases, this map is a measurable Γ -isomorphism.

Remark 1.5. In his Ph.D.-thesis [25, Subsection 7.2.1], Creutz constructs (in the setting described above) finitely supported Furstenberg discretizations on Γ for *certain* compactly supported bi- L -invariant probability measures on H . His construction is however quite different from ours, and does not seem to induce a natural map between the respective Poisson boundaries (this is also not needed for the applications that he had in mind).

We denote by $\text{Prob}(H, L)$ the space of bi- L -invariant probability measures on H , and note that this set is clearly closed under convolution and that the Haar probability measure m_L on L is the neutral element with respect to convolution. In particular, $(\text{Prob}(H, L), *)$ is a monoid. Since L is open in H , every bi- L -invariant measure on H is automatically absolutely continuous with respect to the Haar measure class on H .

Theorem 1.6. *Let (Γ, Λ) be a Hecke pair and (H, L, ρ) a completion triple of (Γ, Λ) . There exists an affine surjective monoid homomorphism*

$$(\text{Prob}(\Gamma; \Lambda), *) \longrightarrow (\text{Prob}(H, L), *), \quad \tau \mapsto \theta_\tau$$

with the following properties:

- (P1) *for every $\tau \in \text{Prob}(\Gamma; \Lambda)$ and right L -invariant $\varphi \in C(H) \cap \mathcal{L}^1(H, \theta_\tau)$,*

$$\sum_{\gamma \in \Gamma} \varphi(\rho(\gamma)) \tau(\gamma) = \int_H \varphi(h) d\theta_\tau(h).$$

In particular, for every θ_τ -integrable homomorphism $\varphi : H \rightarrow \mathbb{R}$, we have

$$\tau(\varphi \circ \rho) = \theta_\tau(\varphi).$$

- (P2) *for every $\tau \in \text{Prob}(\Gamma; \Lambda)$,*

$$\text{supp } \theta_\tau = L\rho(\text{supp } \tau)L.$$

In particular, if τ is spread-out, then so is θ_τ , and if τ has finite support, then θ_τ has compact support.

- (P3) *for every Borel H -space X and L -invariant Borel measure ξ on X ,*

$$\tau * \xi = \theta_\tau * \xi, \quad \text{for all } \tau \in \text{Prob}(\Gamma; \Lambda),$$

where Γ acts on X via ρ . In particular, every (H, θ_τ) -space is also a (Γ, τ) -space.

- (P4) *for every $\tau \in \text{Prob}(\Gamma; \Lambda)$ and Borel (H, θ_τ) -space (X, ξ) ,*

$$\xi \text{ is } \theta_\tau\text{-proximal} \iff \xi \text{ is } \tau\text{-proximal}.$$

Remark 1.7. Property (P2) tells us that spread-out measures are mapped to spread-out measures, so in combination with Property (P3) we can conclude that θ_τ is a Furstenberg discretization of τ whenever τ is spread-out. We stress that we do *not* claim that every compactly supported measure in $\text{Prob}(H, L)$ must necessarily be the image of a finitely supported probability measure on Γ . Property (P4) implies that the Poisson boundary of

(H, θ_τ) , viewed as a (Γ, τ) -space, is a τ -boundary, and thus provides an answer to (A2) in our setting.

Definition 1.8 (Hecke measured group and its Hecke completion). Let (Γ, Λ) be a Hecke pair, and (H, L, ρ) a completion triple of (Γ, Λ) . If τ is a spread-out and Λ -absorbing probability measure on Γ , we say that (Γ, τ) is a *Hecke measured group*, and refer to (H, θ_τ) as the *Hecke completion of (Γ, τ) with respect to ρ* .

Let us now discuss some functorial properties of the map $\tau \mapsto \theta_\tau$ in Theorem 1.6, restricted to spread-out measures. The following corollary will be proved in Section 6, and we retain the notation from Theorem 1.6. We refer to Section 3 and Section 4 for definitions.

Corollary 1.9. *Let τ be a spread-out measure in $\text{Prob}(\Gamma; \Lambda)$, and denote by (B_τ, ν_τ) and $(B_{\theta_\tau}, \nu_{\theta_\tau})$ the Poisson boundaries of (Γ, τ) and (H, θ_τ) respectively, where Γ acts on B_{θ_τ} via ρ .*

- (I) $(B_{\theta_\tau}, \nu_{\theta_\tau})$ is a τ -boundary.
- (II) If M is a compact Γ -space, and η is a Λ -invariant and τ -stationary probability measure on M , then there is a measure-preserving Γ -map

$$(B_{\theta_\tau}, \nu_{\theta_\tau}) \rightarrow (\text{Prob}(M), \eta^*),$$

where $(\text{Prob}(M), \eta^*)$ denotes the canonical quasi-factor of (M, η) . In particular, every τ -proximal and Λ -invariant Borel (Γ, τ) -space is measurably Γ -isomorphic to a θ_τ -boundary, viewed as a (Γ, τ) -space.

- (III) If $(B_{\theta_\tau}, \nu_{\theta_\tau})$ is measurably Γ -isomorphic to (B_τ, ν_τ) , then Λ is amenable.
- (IV) If Λ is amenable and (B_τ, ν_τ) admits a uniquely τ -stationary compact model, then (B_τ, ν_τ) is measurably Γ -isomorphic to $(B_{\theta_\tau}, \nu_{\theta_\tau})$.

1.4. Main lines of investigations

In many situations, Theorem 1.6 and Corollary 1.9 offer a possibility to study certain aspects of (Γ, τ) -spaces through the lenses of (H, θ_τ) -spaces, which are often better behaved (the boundary theory of such measured groups has been systematically developed by Kaimanovich and Woess in [46]).

The benefits of such a change of perspective are especially rewarding when the Poisson boundary of (H, θ_τ) is essentially H -transitive, that is to say, modulo null sets, measurably H -isomorphic to a quotient space of the form H/P for some *closed* subgroup $P < H$, where H/P is endowed with its unique H -invariant measure class. Our main applications (Theorem 1.13 and Theorem 1.19) indeed take place in this scenery (although this point of view is somewhat hidden in the arguments).

Before we describe these applications in detail, we will briefly provide some general background to the lines of investigations that we will pursue in this paper, and to the questions that we will answer.

Existence of prime measured groups

Let (Γ, τ) be a countable measured group and let (X, ξ) be a non-trivial Borel (Γ, τ) -space. We say that (X, ξ) is *prime* if every measure-preserving Γ -map from (X, ξ) to another (Γ, τ) -space is either a measurable Γ -isomorphism or essentially trivial, and we say that a measured group (Γ, τ) is *prime* if the associated Poisson boundary is prime.

As far as we know, before this paper (more specifically, Theorem 1.13 below), not a single example of a prime (countable) measured group was known. Furthermore, the only *explicit* examples in the literature of τ -proximal and prime Borel (Γ, τ) -spaces that we could find (modulo small variations) were:

- Let Γ be a lattice in a simple real Lie group G of real rank at least two, and let Q denote the *maximal* parabolic subgroup of G . Let τ be a spread-out probability measure on Γ such that the (unique) τ -stationary Borel probability measure ν on G/Q is absolutely continuous with respect to the unique G -invariant measure class (the existence of such a measure is guaranteed by Theorem 1.1). Margulis (see e.g. [63, Theorem 8.1.4]) has proved that $(G/Q, \nu)$ is a prime (non-maximal) τ -boundary. This result is a key ingredient in the proof of Margulis' Normal Subgroup Theorem.
- If Γ has Property (T), then Nevo [53, Theorem 4.3] has proved that for every (say, symmetric and finitely supported) spread-out probability measure τ on Γ , the measured group (Γ, τ) always admits (at least one) prime τ -boundary.

Using Corollary 1.9, we can provide (see Section 7) a new class of examples of τ -proximal and prime (Γ, τ) -spaces when Γ is a free group of finite rank.

Corollary 1.10. *There exist an integer $r \geq 2$ and a finitely supported spread-out probability measure τ on a free group Γ of rank r such that the measured group (Γ, τ) admits a prime and essentially free τ -boundary.*

Remark 1.11. In [11], Bourgain constructs a dense free subgroup Γ of finite rank in $SU(1, 1)$ such that the Lebesgue measure ν on the boundary $\partial\mathbb{D}$ of the unit disc \mathbb{D} in the complex plane is the unique τ -stationary probability on $\partial\mathbb{D}$, where τ denotes the uniform probability measure on a set of free generators of Γ . It follows from quite general principles that $(\partial\mathbb{D}, \nu)$ is a τ -boundary, and its primeness and essential freeness can be established along the same lines as in the proof of Corollary 1.10. Similar constructions can also be found in [9] and [10].

Furthermore, Corollary 1.9 also provides the following criterion for when a measured group is *not* prime (see Section 8).

Corollary 1.12. *Let (Γ, Λ) be a Hecke pair, and suppose that Λ is neither amenable nor co-amenable in Γ . Then, for every spread-out and Λ -absorbing probability measure on Γ , the measured group (Γ, τ) is not prime.*

L^p -irreducibility of boundary representations

Let (Γ, τ) be a measured group and let (X, ξ) be a Borel (Γ, τ) -space. For $p \in [1, \infty)$, we denote by $\mathcal{O}(L^p(X, \xi))$ the group of orthogonal transformations on $L^p(X, \xi)$, and we define the map $\sigma_p : \Gamma \rightarrow \mathcal{O}(L^p(X, \xi))$ by

$$\sigma_p(\gamma)f = \left(\frac{d\gamma\xi}{d\xi} \right)^{1/p} f(\gamma^{-1}\cdot), \quad \text{for } \gamma \in \Gamma \text{ and } f \in L^p(X, \xi).$$

One readily checks that σ_p is a homomorphism. We say that (X, ξ) is *L^p -irreducible* if for every non-zero element $f \in L^p(X, \xi)$, the linear span of the set $\{\sigma_p(\gamma)f \mid \gamma \in \Gamma\}$ is norm-dense in $L^p(X, \xi)$. We can of course extend the definition of the L^p -representation σ_p to $p = \infty$ by setting $\sigma_\infty(\gamma)f = f \circ \gamma^{-1}$ for $\gamma \in \Gamma$ and $f \in L^\infty(X, \xi)$. However, since Γ is countable and $L^\infty(X, \xi)$ is non-separable in the norm topology (if the support of ξ is

infinite), (infinite) Borel (Γ, τ) -spaces are never L^∞ -irreducible.

In [6], Bader and Muchnik formulated the following influential conjecture:

Conjecture. *The Poisson boundary of a measured group is L^2 -irreducible.*

Bader and Muchnik [6] proved their conjecture for all Gromov hyperbolic groups, see also [5, 24, 28] for various extensions. We are not aware of any previous investigations into L^p -irreducibility for $p \neq 2$. In Theorem 1.13 below we provide explicit (solvable) measured groups which are *not* L^p -irreducible for any $p \in [1, \infty)$, thereby answering the conjecture above in the negative. The construction of our counterexample depends crucially on the work [42] of Kaimanovich, applied in combination with Theorem 1.6 and Corollary 1.9 above.

The topological structures of boundary entropy spectra

Let (Γ, τ) be a countable measured group and let (X, ξ) be a Borel (Γ, τ) -space. The *Furstenberg entropy* $h_\tau(X, \xi)$ is given by

$$h_\tau(X, \xi) = \sum_{\gamma \in \Gamma} \tau(\gamma) \int_X -\log \frac{d\gamma^{-1}\xi}{d\xi}(x) d\xi(x),$$

whenever the integral is well-defined. The *entropy spectrum* $\text{Ent}(\Gamma, \tau)$ is defined by

$$\text{Ent}(\Gamma, \tau) = \{h_\tau(X, \xi) \mid (X, \xi) \text{ is an ergodic Borel } (\Gamma, \tau)\text{-space}\},$$

and the *boundary entropy spectrum* $\text{BndEnt}(\Gamma, \tau)$ is defined by

$$\text{BndEnt}(\Gamma, \tau) = \{h_\tau(X, \xi) \mid (X, \xi) \text{ is a } \tau\text{-proximal Borel } (\Gamma, \tau)\text{-space}\}.$$

Since every τ -proximal Borel (Γ, τ) -space is ergodic, we have $\text{BndEnt}(\Gamma, \tau) \subseteq \text{Ent}(\Gamma, \tau)$.

Starting with the discussions in [54] by Nevo and Zimmer, the sets $\text{Ent}(\Gamma, \tau)$ and $\text{BndEnt}(\Gamma, \tau)$ have been subject to intense studies. It readily follows from [13] that under mild assumptions, both $\text{Ent}(\Gamma, \tau)$ and $\text{BndEnt}(\Gamma, \tau)$ are continuous images of G_δ -sets, whence analytic subsets of $[0, \infty)$. It has been conjectured (see for instance [19]) that under mild assumptions, $\text{BndEnt}(\Gamma, \tau)$ is always a closed subset.

Bowen proved in [12] that for certain probability measures τ on a free group Γ of finite rank, $\text{Ent}(\Gamma, \tau)$ is a closed interval of the form $[0, h(\tau)]$, where $h(\tau)$ is the Furstenberg entropy of the Poisson boundary of (Γ, τ) . On the other hand, very little is known for $\text{BndEnt}(\Gamma, \tau)$ in the same setting (although Tamuz and Zheng [59] have recently proved that it at least contains a Cantor set).

There are at least two reasons for why the analysis of the boundary entropy spectrum of a measured group is difficult. Firstly, it is often hard to find a manageable parameterization of the set of all τ -proximal Borel (Γ, τ) -spaces. Secondly, even if such a parameterization is available, computing the corresponding Furstenberg entropies is usually quite demanding. Using some results from [7], we shall in Theorem 1.19 below construct from a given prime measured group (for instance one of the ones provided by Theorem 1.13), another measured group, whose boundary entropy spectrum can be computed explicitly. As far as we know, this is the first *explicit* realization of an infinite boundary entropy spectrum of a measured group. We also exhibit a curious phenomenon: a single countable group can admit different spread-out probability measures with radically different boundary entropy spectra; indeed, we construct in Theorem 1.19 below a countable group Γ and two different

spread-out probability measures τ and τ' on Γ so that $\text{BndEnt}(\Gamma, \tau)$ is a Cantor set, while $\text{BndEnt}(\Gamma, \tau')$ is a closed interval.

1.5. Hecke measured groups from non-archimedean local fields

In what follows, let $(K, |\cdot|)$ be a non-archimedean local field with Haar measure m_K . We set

$$\mathcal{O} = \{x \in K \mid |x| \leq 1\} \quad \text{and} \quad \mathcal{P} = \{x \in K \mid |x| < 1\},$$

and assume that the residue field $k = \mathcal{O}/\mathcal{P}$ is a finite cyclic group of (prime) order q . Pick a non-zero element $x_o \in \mathcal{O}$ such that

$$S := \{0, x_o, \dots, (q-1)x_o\} \subset \mathcal{O}$$

is a set of representatives for \mathcal{O}/\mathcal{P} . We fix a uniformizer ϖ of K . Then $|\varpi| = \frac{1}{q}$, and by [56, Proposition 4.17(ii)], every $x \in K$ can be uniquely expressed as a convergent power series in ϖ of the form

$$x = \sum_{j=n}^{\infty} a_j \varpi^j, \quad \text{for some } n \in \mathbb{Z},$$

where $a_j \in S$ for all j . Since K is a field and S has the special form above, we see that the additive group Ξ which is generated by 0 and all powers of ϖ is dense in K . We denote by Ξ_o the subgroup of Ξ which is generated by 0 and all non-negative powers of ϖ , and note that $\Xi_o = \Xi \cap \mathcal{O}$ is dense in \mathcal{O} . One readily checks that

$$\varpi \Xi_o \subset \Xi_o \subset \varpi^{-1} \Xi_o \quad \text{and} \quad |\Xi_o / \varpi \Xi_o| < \infty,$$

which in particular implies that

$$\Gamma = \Xi \rtimes \langle \varpi \rangle \quad \text{and} \quad \Lambda = \Xi_o \rtimes \{1\},$$

where $\langle \varpi \rangle$ denotes the cyclic (multiplicative) group generated by ϖ (which acts on Ξ by multiplication) is a Hecke pair. Furthermore,

$$H = K \rtimes \langle \varpi \rangle \quad \text{and} \quad L = \mathcal{O} \rtimes \{1\} \quad \text{and} \quad \rho = \text{id},$$

is a completion triple of (Γ, Λ) . It is not difficult to show that Γ is a finitely generated group and that H is a lsc compactly generated group. We shall think of Γ as a dense subgroup of H . Moreover, H acts jointly continuously and transitively on K by

$$(x, \varpi^n)y = x + \varpi^n y, \quad \text{for } (x, \varpi^n) \in H \text{ and } y \in K.$$

In particular, $K \cong H/P$, where $P = \{0\} \rtimes \langle \varpi \rangle$. We write $\text{pr}_{\mathbb{Z}}$ for the surjective homomorphism

$$\text{pr}_{\mathbb{Z}} : H \rightarrow \mathbb{Z}, \quad (x, \varpi^n) \mapsto n. \tag{1.1}$$

Our third main theorem now reads as follows.

Theorem 1.13. *Let τ be a finitely supported, spread-out and Λ -absorbing probability measure on Γ . Suppose that*

$$\sum_{n \in \mathbb{Z}} n (\text{pr}_{\mathbb{Z}})_* \tau(n) < 0. \tag{1.2}$$

Then the following holds.

- (I) *There exists a unique τ -stationary probability measure ν on K .*
- (II) *ν is \mathcal{O} -invariant, H -quasi-invariant and absolutely continuous with respect to m_K with an \mathcal{O} -invariant (thus continuous) everywhere positive density.*
- (III) *The Borel (Γ, τ) -space (K, ν) is prime and τ -proximal.*

(IV) For every $p \in [1, \infty)$, the quasi-regular Γ -representation on $L^p(K, \nu)$ is not irreducible.

Remark 1.14. The existence, uniqueness and τ -proximality of ν (assuming that τ is finitely supported, spread-out and satisfies 1.2) is a standard result (see e.g. [42, 16, 17]). Furthermore, (III) and (IV) hold whenever ν is absolutely continuous with respect to the Haar measure m_K . Only (II), which is the key point of the theorem, exploits the assumption that τ is Λ -absorbing. This raises the question how essential Λ -absorption is to ensure that ν is absolutely continuous with respect to m_K . In general this is quite a subtle question (for archimedean fields, this is closely related to the classical line of research, initiated by Erdős and Wintner, pertaining to absolute continuity of Bernoulli convolutions). Recently, Briussel and Tanaka [15] have developed a technique to construct finitely supported and spread-out probability measures on groups of *real* affine transformations, whose actions on the real line admit unique stationary measures which are *singular* with respect to the Lebesgue measure. We plan to extend their techniques to non-archimedean fields in future works, thus showing that Λ -absorption of τ is crucial to establish (II).

Remark 1.15. Before we proceed, we show how one can explicitly construct finitely supported and spread-out τ in $\text{Prob}(\Gamma; \Lambda)$ which satisfy (1.2). Fix $0 < \delta < 1/2$. Given two probability measures τ_1 and τ_{-1} on Ξ , we define

$$\tau(x, \varpi^n) = \begin{cases} \tau_1(x)\delta & \text{if } n = 1 \\ \tau_{-1}(x)(1 - \delta) & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{for } (x, \varpi^n) \in \Gamma.$$

We note that τ is a probability measure on Γ for which (1.2) holds. It thus remains to produce finitely supported probability measures τ_1 and τ_{-1} on Ξ so that τ is spread-out and Λ -absorbing. We first observe that

$$\alpha_*\tau((x, \varpi^n)\Lambda) = \begin{cases} \left(\sum_{\xi \in \Xi_o} \tau_1(x + \varpi\xi)\right)\delta & \text{if } n = 1 \\ \left(\sum_{\xi \in \Xi_o} \tau_{-1}(x + \varpi^{-1}\xi)\right)(1 - \delta) & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{for } (x, \varpi^n)\Lambda \in \Gamma/\Lambda.$$

The third expression is clearly Λ -invariant, and since $\varpi^{-1}\Xi_o \supset \Xi_o$, the middle expression is Λ -invariant as well, for any choice of τ_{-1} . Hence we must only construct τ_1 so that the first expression is Λ -invariant. To do this, we pick a set of representatives T for $\Xi_o/\varpi\Xi_o$. Then, for every probability measure κ on Ξ , we set

$$\tau_1(x) = \frac{1}{|T|} \sum_{t \in T} \kappa(x + t), \quad \text{for } x \in \Xi,$$

and note that τ_1 is a probability measure on Ξ , and

$$\sum_{\xi \in \Xi_o} \tau_1(x + \varpi\xi) = \frac{1}{|T|} \sum_{t \in T} \sum_{\xi \in \varpi\Xi_o} \kappa(x + t + \xi) = \frac{1}{|T|} \sum_{\xi \in \Xi_o} \kappa(x + \xi),$$

which is clearly Ξ_o -invariant, whence τ with this choice of τ_1 will be Λ -absorbing. If we further ensure that τ_{-1} and κ are finitely supported, so that the resulting support of τ contains a finite generating set for Γ , then we have constructed a finitely supported, spread-out and Λ -absorbing probability measure on Γ which satisfies (1.2).

The question arises whether (K, ν) in Theorem 1.13 is the Poisson boundary of (Γ, τ) . Since we are not aware of a good reference in this complete generality, we confine our attention to two important special cases, to which the work [42] of Kaimanovich applies.

Example 1.1 (Solvable Baumslag-Solitar groups). Let q be a prime number, and set

$$K = \mathbb{Q}_q \quad \text{and} \quad S = \{0, 1, \dots, q-1\} \quad \text{and} \quad \varpi = q.$$

Then $\Xi \cong \mathbb{Z}[1/q]$ and $\Xi_o \cong \mathbb{Z}$. It is not difficult to see that Γ is isomorphic to the Baumslag-Solitar group $\text{BS}(1, q) = \langle a, b \mid bab^{-1} = a^q \rangle$, with $\Lambda \cong \langle a \rangle \cong \mathbb{Z}$.

Example 1.2 (Lamplighter groups). Let q be a prime number, and set

$$K = \mathbb{F}_q(t) \quad \text{and} \quad S = \{0, 1, \dots, q-1\} \quad \text{and} \quad \varpi = t,$$

where \mathbb{F}_q denotes the finite field with q elements. Then $\Xi \cong \bigoplus_{\mathbb{Z}} \mathbb{F}_q$ and $\Xi_o \cong \bigoplus_{\mathbb{N}} \mathbb{F}_q$. It is not difficult to see that Γ is isomorphic to the wreath product (lamplighter group) $\mathbb{F}_q \wr \mathbb{Z}$. Note that in this case, Λ is an infinite locally finite group, whence infinitely generated.

Kaimanovich [42, Section 5 and 6] (see also Brofferio [16, 17]) has shown that in the two examples above, (K, ν) is in fact the Poisson boundary of (Γ, τ) , if τ is finitely supported and satisfies (1.2). In particular, we now have the following corollary.

Corollary 1.16. *Let Γ be as in either Example 1.1 or Example 1.2, and let τ be as in Theorem 1.13. Then (Γ, τ) is a prime measured group, whose Poisson boundary is not L^p -irreducible for any $p \in [1, \infty]$.*

1.6. Explicit realizations of boundary entropy spectra

Takeya suggested in [47] an interesting way to generate closed subsets of the real line along the following lines. Given a positive summable sequence $\beta = (\beta_1, \beta_2, \dots)$, define its *subsum set* $\text{SubSum}(\beta)$ by

$$\text{SubSum}(\beta) = \left\{ \sum_{n \in S} \beta_n \mid S \subseteq \mathbb{N} \right\} \subset [0, \infty).$$

It is not difficult to show that $\text{SubSum}(\beta)$ is always a closed set, and Takeya proved in 1915 that it is also perfect. After subsequent independent work by Hornich [38] and Guthrie and Nymann [36] (see also the survey [55]), the structure of subsum sets is now very well understood. We summarize their findings in the following theorem.

Theorem 1.17. *Let $\beta = (\beta_1, \beta_2, \dots)$ be a positive summable sequence.*

- (I) $\text{SubSum}(\beta)$ is either:
- a finite union of disjoint closed intervals.
 - a Cantor set.
 - a symmetric Cantorval, i.e. a non-empty compact subset of $[0, \infty)$, which is equal to the closure of its interior, and which has the property that every pair of endpoints of a non-trivial connected component, consists of accumulation points of one point components.
- (II) Suppose that β is non-increasing, and set $B_n = \sum_{k > n} \beta_k$ for $n \geq 0$.
- If $\beta_n > B_n$ for all $n \geq 1$, then $\text{SubSum}(\beta)$ is a Cantor set with Lebesgue measure $\lim_n 2^n B_n$.
 - If $\beta_n \leq B_n$ for all $n \geq 1$, then $\text{SubSum}(\beta)$ is the interval $[0, B_0]$.

Remark 1.18. In particular, if we let $\beta_n = a\rho^{n-1}$ for some $a > 0$ and $\rho \in (0, 1)$, then

$$\text{SubSum}(\beta) = \begin{cases} \text{a Cantor set of Lebesgue measure zero if } \rho < 1/2. \\ \text{the interval } [0, \frac{a}{1-\rho}] \text{ if } \rho \geq 1/2. \end{cases}$$

Our next theorem connects subsum sets with boundary entropy spectra of measured groups. We recall from the discussion above that the boundary entropy spectrum of a measured group (Γ, τ) is given by

$$\text{BndEnt}(\Gamma, \tau) = \{h_\tau(X, \xi) \mid (X, \xi) \text{ is a } \tau\text{-proximal Borel } (\Gamma, \tau)\text{-space}\}.$$

Theorem 1.19. *There exists a countable discrete group Γ with the following property: for every positive summable sequence $\beta = (\beta_1, \beta_2, \dots)$, there is a spread-out probability measure τ_β on Γ such that*

$$\text{BndEnt}(\Gamma, \tau_\beta) = \text{SubSum}(\beta).$$

In particular we can find spread-out probability measures τ and τ' on Γ such that the boundary entropy spectrum of (Γ, τ) is a Cantor set, while the boundary entropy spectrum of (Γ, τ') is an interval.

Remark 1.20. It follows from the proof of Theorem 1.19, in combination with Corollary 1.16, that we can for instance take $\Gamma = \bigoplus_{\mathbb{N}} \text{BS}(1, 2)$, the direct sum of countably many copies of the Baumslag-Solitar group $\text{BS}(1, 2)$. We do not know if Γ in Theorem 1.19 can be chosen *finitely generated*.

1.7. Some questions

In this subsection we briefly collect some questions and open problems which are related to the theorems, discussions and corollaries above.

In what follows, let (Γ, Λ) be a Hecke pair, (H, L, ρ) a completion triple of (Γ, Λ) and τ a Λ -absorbing and spread-out probability measure on Γ . Let (H, θ_τ) denote the Hecke completion of (Γ, τ) with respect to ρ , and write (B_τ, ν_τ) and $(B_{\theta_\tau}, \nu_{\theta_\tau})$ for the Poisson boundaries of the measured groups (Γ, τ) and (H, θ_τ) respectively.

MARTIN BOUNDARIES

Question 1. Is there a natural relation between the (minimal) Martin boundary of (Γ, τ) and the (minimal) Martin boundary of (H, θ_τ) ?

A RELATIVE LIOUVILLE THEOREM FOR HECKE PAIRS

The following question asks for a converse to (III) in Corollary 1.9.

Question 2. Suppose that Λ is amenable. Given a bi- L -invariant spread-out probability measure θ on H , can we find a spread-out Λ -absorbing probability measure τ on Γ such that $\theta_\tau = \theta$ and the Poisson boundaries of (Γ, Λ) and (H, θ) are measurably Γ -isomorphic?

If Λ is a *normal* subgroup of Γ (in which case *every* measure on Γ is Λ -absorbing), Kaimanovich [44, Theorem 1] has answered this question in the affirmative. The special case when $\Gamma = \Lambda$ was an influential conjecture of Furstenberg, and was solved independently by Kaimanovich and Vershik in [45] and by Rosenblatt in [57].

WHEN IS $(B_{\theta_\tau}, \nu_{\theta_\tau})$ A TRANSITIVE BOREL H -SPACE?

In Subsection 4.5 (cf. Corollary 4.15) we show that the study of τ -boundaries simplifies significantly if one knows that the H -action on B_{θ_τ} is $(\nu_{\theta_\tau}$ -essentially) transitive. Various

criteria for when the Poisson boundary of a measured group is transitive was developed by Azencott [4] for semisimple Lie groups, and by Jaworski [40, 41] for almost connected locally compact groups. On the other hand, we are only aware of two general criteria which (also) cover *totally disconnected* groups. We briefly summarize these criteria:

- **(H, L) is a Gelfand pair** (the convolution algebra of compactly supported bi- L -invariant functions on H is commutative). In this case, the compact group L acts ergodically on $(B_{\theta_\tau}, \nu_{\theta_\tau})$, and thus transitively (by compactness of L), whence H acts transitively as well (The fact that L acts ergodically follows from the arguments of Monod in [51], see also [35, Chaper XII] for a related approach). An illustrative example of a Gelfand pair is

$$H = \mathrm{SL}_2(\mathbb{Q}_p) \quad \text{and} \quad L = \mathrm{SL}_2(\mathbb{Z}_p), \quad \text{for a prime number } p.$$

We stress that it is not important in this criterion that H is totally disconnected.

- **H fixes one of infinitely many ends.** Suppose that H is compactly generated, and that for a fixed compact generating set $S \subset H$, the Schreier graph $\mathcal{X} = \mathcal{X}(H, L, S)$ associated with triple (H, L, S) has infinitely many ends (the vertices of this graph are H/L , and two vertices xL and yL are connected by an edge if the intersection $Lx^{-1}yL \cap S$ is non-empty). Let $\partial\mathcal{X}$ denote the space of ends of \mathcal{X} , and suppose that H fixes an end ω . Then, by [52], H is non-unimodular, amenable and acts transitively on $\partial\mathcal{X} \setminus \{\omega\}$. Furthermore, if θ_τ is compactly supported and spread-out, and

$$\int_H \log \Delta_H d\theta_\tau > 0,$$

where Δ_H denotes the modular function on H , then [46, Theorem 6.12b)] asserts that there exists a (unique) θ_τ -stationary probability measure ν on $\partial\mathcal{X} \setminus \{\omega\}$ such that $(\partial\mathcal{X} \setminus \{\omega\}, \nu)$ is the Poisson boundary of (H, θ_τ) .

Question 3. Are there other criteria for when $(B_{\theta_\tau}, \nu_{\theta_\tau})$ is a transitive Borel H -space?

NON-AMENABLE BAUMSLAG-SOLITAR GROUPS

The Baumslag-Solitar group $\mathrm{BS}(p, q)$ is defined by

$$\mathrm{BS}(p, q) = \langle a, b \mid ab^p a^{-1} = b^q \rangle, \quad \text{for non-zero integers } p, q.$$

One readily checks that $(\mathrm{BS}(p, q), \langle b \rangle)$ is a Hecke pair, and an explicit completion triple $(H_{p,q}, L_{p,q}, \rho_{p,q})$ of this Hecke pair was first produced by Gal and Januszkiewicz in [34].

Let us fix integers p and q with $2 \leq |p| < |q|$, so that $\mathrm{BS}(p, q)$ is non-amenable, and let τ be a finitely supported $\langle b \rangle$ -absorbing probability measure on $\mathrm{BS}(p, q)$. Under mild assumptions, Cuno and Sava-Huss [22] have proved that the space of ends $\Omega_{T_{p,q}}$ of the Bass-Serre tree $T_{p,q}$ associated with $(\mathrm{BS}(p, q), \langle b \rangle)$ is a compact and uniquely τ -stationary model for the Poisson boundary of $(\mathrm{BS}(p, q), \tau)$. Let ν denote the unique τ -stationary probability measure on $\Omega_{T_{p,q}}$. Since $\langle b \rangle \cong \mathbb{Z}$ is amenable, (IV) in Corollary 1.9 tells us that the Poisson boundary of $(H_{p,q}, \theta_\tau)$ is measurably Γ -isomorphic to $(\Omega_{T_{p,q}}, \nu)$, and thus every τ -boundary is measurably Γ -isomorphic to a θ_τ -boundary, viewed as a Γ -space.

Question 4. Is there a "reasonable" parameterization of the set of θ_τ -boundaries in this setting?

ANANTHARAMAN-DELAROCHE GROUPS

In the recent paper [3], Anantharaman-Delaroche constructs Hecke pairs (Γ, Λ) and completion triples (H, L, ρ) such that

- (i) Λ , and thus Γ , are non-amenable finitely generated groups.
- (ii) Λ is co-amenable (and thus H is amenable), compactly generated and admits a transitive action on a tree (with compact stabilizers).
- (iii) ρ is injective.

Let us fix such a Hecke pair (Γ, Λ) and an associated completion triple (H, L, ρ) , and let θ be a *symmetric*, bi- L -invariant, compactly generated and spread-out probability measure on H . By Theorem 1.6, there exists at least one Λ -absorbing probability measure τ on Γ (which does not need to be symmetric) such that $\theta = \theta_\tau$. Since Γ is non-amenable, the Poisson boundary (B_τ, ν_τ) must be non-trivial. However, by [46, Theorem 6.12a)], the symmetricity of θ_τ forces the Poisson boundary of (H, θ_τ) to be trivial.

Question 5. Is (B_τ, ν_τ) a prime (Γ, τ) -space (cf. Corollary 1.12)?

HIGMAN-THOMPSON'S GROUP AND THE NERETIN GROUP

We refer to [20] for definitions. Let $d \geq 2$ and $k \geq 1$, and denote by $\mathcal{T}_{d,k}$ the unique rooted tree having k vertices of level 1, each of which is attached to an underlying d -regular tree. We write $\Gamma_{d,k}$ and $H_{d,k}$ for the Higman-Thompson group and the Neretin group associated with $\mathcal{T}_{d,k}$ respectively. It is known that there is a injective homomorphism ρ from $\Gamma_{d,k}$ into $H_{d,k}$ with a dense image, as well as a compact and open subgroup $L_{d,k}$ such that $\Lambda_{d,k} := \rho^{-1}(L_{d,k})$ is a locally finite (and thus amenable) subgroup of $\Gamma_{d,k}$. The following two questions seem to arise naturally.

Question 6. Suppose that τ is a finitely supported and spread-out $\Lambda_{d,k}$ -absorbing probability measure on $\Gamma_{d,k}$. Does the Poisson boundary of $(\Gamma_{d,k}, \tau)$ admit a uniquely τ -stationary compact model (cf. (IV) in Corollary 1.9)?

Question 7. Is the Poisson boundary of $(H_{d,k}, \theta_\tau)$ a transitive $H_{d,k}$ -space?

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2. PROOF OF THEOREM 1.4

Let Γ be a countable group Λ a subgroup of Γ , and write $\alpha : \Gamma \rightarrow \Gamma/\Lambda$ for the canonical quotient map. The induced map $\alpha_* : \text{Prob}(\Gamma) \rightarrow \text{Prob}(\Gamma/\Lambda)$ is given by

$$\bar{\tau}(\gamma\Lambda) := \alpha_*\tau(\gamma\Lambda) = \sum_{\lambda \in \Lambda} \tau(\gamma\lambda), \quad \text{for } \gamma\Lambda \in \Gamma/\Lambda. \quad (2.1)$$

We set $\text{Prob}(\Gamma; \Lambda) = \{\tau \in \text{Prob}(\Gamma) \mid \bar{\tau} \text{ is } \Lambda\text{-invariant}\}$.

Proof of (I)

Fix $\tau_1, \tau_2 \in \text{Prob}(\Gamma; \Lambda)$, and note that

$$\begin{aligned}
\alpha_*(\tau_1 * \tau_2)(\gamma\Lambda) &= \sum_{\lambda \in \Lambda} (\tau_1 * \tau_2)(\gamma\lambda) = \sum_{\eta \in \Gamma} \sum_{\lambda \in \Lambda} \tau_1(\eta) \tau_2(\eta^{-1}\gamma\lambda) \\
&= \sum_{\eta \in \Gamma} \tau_1(\eta) \bar{\tau}_2(\eta^{-1}\gamma\Lambda) = \sum_{\eta\Lambda} \sum_{\lambda' \in \Lambda} \tau_1(\eta\lambda') \bar{\tau}_2(\eta^{-1}\gamma\Lambda) \\
&= \sum_{\eta\Lambda} \bar{\tau}_1(\eta\Lambda) \bar{\tau}_2(\eta^{-1}\gamma\Lambda), \quad \text{for } \gamma\Lambda \in \Gamma/\Lambda,
\end{aligned} \tag{2.2}$$

where we in the second to last step have used that $\bar{\tau}_2$ is Λ -invariant. Since $\bar{\tau}_1$ is Λ -invariant, we conclude that $\alpha_*(\tau_1 * \tau_2)$ is Λ -invariant. In particular, $\tau_1 * \tau_2 \in \text{Prob}(\Gamma; \Lambda)$ as well.

Proof of (II)

Suppose that Γ acts by linear isometries on a Banach space E and that the dual action preserves a weak*-closed convex subset C of E^* . If C^Λ is empty, there is nothing to prove, so let us assume that $C^\Lambda \neq \emptyset$, and pick $c \in C^\Lambda$. Then, for every $\tau \in \text{Prob}(\Gamma)$,

$$\tau * c = \sum_{\gamma \in \Gamma} \tau(\gamma) \gamma c = \sum_{\gamma\Lambda} \sum_{\lambda \in \Lambda} \tau(\gamma\lambda) \gamma\lambda c = \sum_{\gamma\Lambda} \bar{\tau}(\gamma\Lambda) \gamma c.$$

If τ is Λ -absorbing, then for every $\lambda \in \Lambda$

$$\begin{aligned}
\lambda(\tau * c) &= \sum_{\gamma\Lambda} \bar{\tau}(\gamma\Lambda) \lambda\gamma c = \sum_{\gamma\Lambda} \bar{\tau}(\lambda\gamma\Lambda) \lambda\gamma c \\
&= \sum_{\gamma\Lambda} \bar{\tau}(\gamma\Lambda) \gamma c = \tau * c,
\end{aligned}$$

whence $\tau * C^\Lambda \subseteq C^\Lambda$.

Proof of (III)

Fix a right-inverse $\beta : \Gamma/\Lambda \rightarrow \Gamma$ for the map α . Then, for every $\gamma \in \Gamma$, we can write

$$\gamma = \beta(\gamma\Lambda) \lambda_\gamma,$$

for some *unique* $\lambda_\gamma \in \Lambda$. In particular, given a probability measure r on Λ , we can define an (affine) right-inverse

$$\text{Prob}(\Gamma/\Lambda) \rightarrow \text{Prob}(\Gamma), \quad \hat{\tau} \mapsto \tilde{\tau}$$

of the map α_* by setting

$$\tilde{\tau}(\gamma) = \hat{\tau}(\gamma\Lambda) r(\lambda_\gamma), \quad \text{for } \gamma \in \Gamma.$$

Note that

$$\text{supp } \tilde{\tau} = \{\gamma \in \Gamma \mid \alpha(\gamma) \in \text{supp } \hat{\tau}, \lambda_\gamma \in \text{supp } r\}.$$

Let us now assume that (Γ, Λ) is a Hecke pair. Then, since all Λ -orbits in Γ/Λ are finite, we get a natural affine retraction

$$\text{Prob}(\Gamma/\Lambda) \rightarrow \text{Prob}(\Gamma/\Lambda)^\Lambda, \quad \bar{\tau} \mapsto \hat{\tau}$$

upon averaging over the Λ -orbits. Clearly, $\text{supp } \hat{\tau} = \Lambda \cdot \text{supp}(\bar{\tau})$. Consider the (affine) composition

$$\text{Prob}(\Gamma) \rightarrow \text{Prob}(\Gamma; \Lambda), \quad \tau \mapsto \tilde{\tau},$$

given by $\tau \mapsto \bar{\tau} := \alpha_*\tau \mapsto \hat{\tau} \mapsto \tilde{\tau}$. Then, since the support of $\bar{\tau}$ is $\alpha(\text{supp } \tau)$, we see that

$$\text{supp } \tilde{\tau} = \{\gamma \in \Gamma \mid \gamma\Lambda \in \Lambda.\alpha(\text{supp } \tau), \lambda_\gamma \in \text{supp } r\} \quad (2.3)$$

$$\subseteq \beta(\Lambda.\alpha(\text{supp } \tau)) \text{supp } r \quad (2.4)$$

Fix a subset S of Γ , and define

$$S_\Lambda = \{\lambda_\gamma \in \Lambda \mid \gamma \in S\}.$$

Let us assume that the support of the probability measure r above equals S_Λ , and set

$$\tilde{S} = \beta(\Lambda.\alpha(S))S_\Lambda \subset \Gamma.$$

If S is finite, then S_Λ and \tilde{S} are finite sets. It follows from (2.4) that the map $\tau \mapsto \tilde{\tau}$ restricts to an affine map

$$\text{Prob}(S) \rightarrow \text{Prob}(\Gamma; \Lambda) \cap \text{Prob}(\tilde{S}).$$

Furthermore, if $\gamma \in \text{supp } \tau \subset S$, then clearly

$$\gamma\Lambda \in \alpha(\text{supp } \tau) \subset \Lambda.\alpha(\text{supp } \tau) \quad \text{and} \quad \lambda_\gamma \in S_\Lambda,$$

whence $\gamma \in \text{supp } \tilde{\tau}$ by (2.3) and since r has full support on S_Λ . Since γ is arbitrary, we conclude that $\text{supp } \tau \subseteq \text{supp } \tilde{\tau}$.

Remark 2.1. If we assume that for every $\gamma \in \Gamma$, all elements of Λ and $\gamma\Lambda\gamma^{-1}$ commute with each other (this is for instance the case for the Hecke pairs in Theorem 1.13), then there is another affine construction of Λ -absorbing probability measures on Γ which can be constructed along the following lines. For every $\gamma \in \Gamma$, we choose sets of representatives $A_{\gamma\Lambda}$ and $B_{\gamma\Lambda}$ for the right- and left-quotients

$$\Lambda/\Lambda \cap \gamma\Lambda\gamma^{-1} \quad \text{and} \quad \Lambda \cap \gamma\Lambda\gamma^{-1} \backslash \gamma\Lambda\gamma^{-1}$$

respectively. Since all elements of Λ and $\gamma\Lambda\gamma^{-1}$ are assumed to commute with each other, and thus

$$\lambda\gamma\Lambda\gamma^{-1}\lambda^{-1} = \gamma\Lambda\gamma^{-1}, \quad \text{for all } \gamma \in \Gamma \text{ and } \lambda \in \Lambda,$$

we may assume that the maps $\gamma \mapsto A_{\gamma\Lambda}$ and $\gamma \mapsto B_{\gamma\Lambda}$ are left- Λ -invariant.

Given $\tau \in \text{Prob}(\Gamma)$, we set

$$\tilde{\tau}(\gamma) = \frac{1}{|A_{\gamma\Lambda}|} \sum_{a \in A_{\gamma\Lambda}} \tau(a\gamma), \quad \text{for } \gamma \in \Gamma,$$

and note that $\tau \mapsto \tilde{\tau}$ is affine, and

$$\begin{aligned} \alpha_*\tilde{\tau}(\gamma\Lambda) &= \sum_{\lambda \in \Lambda} \tilde{\tau}(\gamma\lambda) = \frac{1}{|A_{\gamma\Lambda}|} \sum_{a \in A_{\gamma\Lambda}} \sum_{\lambda \in \Lambda} \tau(a\gamma\lambda) = \frac{1}{|A_{\gamma\Lambda}|} \sum_{a \in A_{\gamma\Lambda}} \sum_{\lambda \in \gamma\Lambda\gamma^{-1}} \tau(a\lambda\gamma) \\ &= \frac{1}{|A_{\gamma\Lambda}|} \sum_{b \in B_{\gamma\Lambda}} \sum_{a \in A_{\gamma\Lambda}} \sum_{\lambda \in \Lambda \cap \gamma\Lambda\gamma^{-1}} \tau(a\lambda b\gamma) \\ &= \frac{1}{|A_{\gamma\Lambda}|} \sum_{b \in B_{\gamma\Lambda}} \sum_{\lambda \in \Lambda} \tau(\lambda b\gamma). \end{aligned}$$

Since $B_{\gamma\Lambda} \subset \gamma\Lambda\gamma^{-1}$ and all elements of Λ and $\gamma\Lambda\gamma^{-1}$ commute with each other, we have

$$\alpha_*\tilde{\tau}(\gamma\Lambda) = \frac{1}{|A_{\gamma\Lambda}|} \sum_{b \in B_{\gamma\Lambda}} \sum_{\lambda \in \Lambda} \tau(b\lambda\gamma), \quad \text{for all } \gamma\Lambda \in \Gamma/\Lambda,$$

which is clearly a left- Λ -invariant expression since the maps $\gamma \mapsto A_{\gamma\Lambda}$ and $\gamma \mapsto B_{\gamma\Lambda}$ are left- Λ -invariant.

3. PRELIMINARIES ON COMPACT G -SPACES WITH μ -STATIONARY MEASURES

Let G be a locally compact and second countable (lcsc) group and let μ be a probability measure on G . Let M be a compact and metrizable space, and denote by $\text{Prob}(M)$ the space of Borel probability measures on M , equipped with the (compact and metrizable) weak*-topology. If G acts jointly continuously by homeomorphisms on M , then we say that M is a *compact G -space*. In this case, G also acts jointly continuously by homomorphisms on $\text{Prob}(M)$, and we say that $\eta \in \text{Prob}(M)$ is μ -stationary if $\mu * \eta = \eta$.

We denote by $\text{Prob}_\mu(M)$ the set of all μ -stationary probability measures on M , and write $\text{Prob}_\mu^{\text{erg}}(M)$ and $\text{Prob}_\mu^{\text{ext}}(M)$ for the subsets of ergodic and extremal measures in $\text{Prob}_\mu(M)$ respectively.

Lemma 3.1. *Let μ be a spread-out probability measure on G and let M be a compact G -space.*

- (I) *Every μ -stationary probability measure on M is G -quasi-invariant.*
- (II) $\text{Prob}_\mu^{\text{erg}}(M) = \text{Prob}_\mu^{\text{ext}}(M) \neq \emptyset$.
- (III) *If η and η' are ergodic μ -stationary probability measures and if η' is absolutely continuous with respect to η , then $\eta = \eta'$.*

Proof. (II) is proved in [54, Lemma 1.1]. The identity in (I) is contained in [7, Corollary 2.7], while the assertion of non-emptiness is an immediate consequence of Kakutani's fixed point theorem. (III) is proved in [7, Proposition 2.6:(2)]. \square

Corollary 3.2. *Let μ_1 and μ_2 be spread-out probability measures on G and let M be a compact G -space. Suppose that $\mu_1 * \mu_2 = \mu_2 * \mu_1$. Then $\text{Prob}_{\mu_1}(M) = \text{Prob}_{\mu_2}(M)$.*

Proof. By (II) in Lemma 3.1 it suffices to show that $\text{Prob}_{\mu_1}^{\text{erg}}(M) \subset \text{Prob}_{\mu_2}(M)$. To prove this inclusion, fix an ergodic $\eta \in \text{Prob}_{\mu_1}(M)$ and set $\eta' := \mu_2 * \eta$. Since μ_1 is spread-out, η is G -quasi-invariant by (I) in Lemma 3.1, and thus $\eta' \ll \eta$. Note that since $\mu_1 * \mu_2 = \mu_2 * \mu_1$,

$$\mu_1 * \eta' = \mu_1 * \mu_2 * \eta = \mu_2 * \mu_1 * \eta = \mu_2 * \eta = \eta',$$

whence η' is μ_1 -stationary and absolutely continuous with respect to η , and thus $\eta' = \eta$ by (III) in Lemma 3.1. This shows that $\eta \in \text{Prob}_{\mu_2}(M)$, and we are done. \square

3.1. Conditional measures and the canonical quasi-factor

Lemma 3.3. [7, Theorem 2.10] *Let μ be a probability measure on G , M a compact G -space and η a μ -stationary probability measure on M . There is a $\mu^{\mathbb{N}}$ -conull subset $\Omega_\eta \subset G^{\mathbb{N}}$ and a measurable map*

$$\Omega \rightarrow \text{Prob}(M), \quad \omega \mapsto \eta_\omega,$$

such that $\lim_n \omega_1 \cdots \omega_n \eta = \eta_\omega$, for all $\omega = (\omega_1, \omega_2, \dots) \in \Omega_\eta$, where the limit is taken in the weak-topology on $\text{Prob}(M)$.*

Remark 3.4. We stress that we do not assume that μ is a spread-out probability measure on G .

Definition 3.5 (Conditional measures). The map $\omega \mapsto \eta_\omega$ in Lemma 3.3 is called the *conditional measure map* associated with (M, η) , and the measures (η_ω) are called *conditional measures*.

In what follows, let μ be a probability measure on G and let (M, η) be a compact (G, μ) -space. Given a bounded Borel function f on M , we define

$$\widehat{f}(\beta) = \beta(f), \quad \text{for } \beta \in \text{Prob}(M).$$

We note that if f is continuous (Borel measurable), then \widehat{f} is continuous (Borel measurable) with respect to the weak*-topology on $\text{Prob}(M)$. Moreover,

$$\int_M f d\eta = \int_{G^{\mathbb{N}}} \widehat{f}(\eta_\omega) d\mu^{\otimes \mathbb{N}}(\omega), \quad \text{for all } f \in C(M), \quad (3.1)$$

where $\omega \mapsto \eta_\omega$ denotes the conditional measure map associated with (M, η) . A straightforward approximation argument in $L^1(\eta)$ also shows that (3.1) holds for every bounded Borel function on M .

Definition 3.6 (The canonical quasi-factor). The probability measure η^* on $\text{Prob}(M)$ defined by

$$\eta^*(F) = \int_{G^{\mathbb{N}}} F(\eta_\omega) d\mu^{\otimes \mathbb{N}}(\omega), \quad \text{for } F \in C(\text{Prob}(M)),$$

where $\omega \mapsto \eta_\omega$ is the conditional measure map associated with (M, η) , is called the *canonical quasi-factor* of (M, η) .

Remark 3.7. We note that (3.1) says that $\eta^*(\widehat{f}) = \eta(f)$ for all $f \in C(M)$, which in particular implies that $(\text{Prob}(M), \eta^*)$ is a μ -stationary quasi-factor of (M, η) in the sense of Furstenberg and Glasner (see e.g. [33, Section 1]).

It is not hard to see that the set of all \widehat{f} , as f ranges over $C(M)$, separates points in $\text{Prob}(M)$, whence the *-algebra generated by such functions is dense in $C(\text{Prob}(M))$ by Stone-Weierstrass Theorem. In particular, η^* is completely determined by all expressions of the form $\eta^*(\widehat{f}_1 \cdots \widehat{f}_k)$, where $k \geq 1$ and f_1, \dots, f_k is a k -tuple in $C(M)$. The following lemma provides a technique to evaluate these expression.

Lemma 3.8. For every $k \geq 1$, and for all bounded Borel functions f_1, \dots, f_k on M ,

$$\eta^*(\widehat{f}_1 \cdots \widehat{f}_k) = \lim_{n \rightarrow \infty} (\mu^{*n} * \eta^{\otimes k})(f_1 \otimes \cdots \otimes f_k).$$

Proof. Fix $k \geq 1$. Let us first consider the case when f_1, \dots, f_k are continuous. We note that

$$\begin{aligned} (\mu^{*n} * \eta^{\otimes k})(f_1 \otimes \cdots \otimes f_k) &= \int_G g\eta(f_1) \cdots g\eta(f_k) d\mu^{*n}(g) \\ &= \int_{G^{\mathbb{N}}} (\omega_1 \cdots \omega_n \eta)(f_1) \cdots (\omega_1 \cdots \omega_n \eta)(f_k) d\mu^{\mathbb{N}}(\omega). \end{aligned}$$

By Lemma 3.3,

$$\eta_\omega(f) = \lim_{n \rightarrow \infty} (\omega_1 \cdots \omega_n \eta)(f), \quad \text{for all } f \in C(M),$$

$\mu^{\mathbb{N}}$ -almost surely, whence by dominated convergence

$$\lim_{n \rightarrow \infty} \int_{G^{\mathbb{N}}} (\omega_1 \cdots \omega_n \eta)(f_1) \cdots (\omega_1 \cdots \omega_n \eta)(f_k) d\mu^{\mathbb{N}}(\omega) = \int_{G^{\mathbb{N}}} \eta_\omega(f_1) \cdots \eta_\omega(f_k) d\mu^{\mathbb{N}}(\omega).$$

We now note that

$$\int_{G^{\mathbb{N}}} \eta_\omega(f_1) \cdots \eta_\omega(f_k) d\mu^{\mathbb{N}}(\omega) = \int_{G^{\mathbb{N}}} \widehat{f}_1(\eta_\omega) \cdots \widehat{f}_k(\eta_\omega) d\mu^{\mathbb{N}}(\omega) = \eta^*(\widehat{f}_1 \cdots \widehat{f}_k),$$

which proves the lemma for continuous functions.

To prove the lemma in general, pick a k -tuple (f_1, \dots, f_k) of bounded Borel functions on M . For every $\varepsilon > 0$, we can find a k -tuple $(f_{1,\varepsilon}, \dots, f_{k,\varepsilon})$ of continuous functions on M such that

$$\|f_{i,\varepsilon}\|_\infty \leq \|f_i\|_\infty \quad \text{and} \quad \|f_i - f_{i,\varepsilon}\|_{L^1(\eta)} < \varepsilon, \quad \text{for all } i = 1, \dots, k.$$

We write

$$\begin{aligned} \eta^*(\widehat{f}_1 \cdots \widehat{f}_k) &= \eta^*(\widehat{f}_{1,\varepsilon} \cdots \widehat{f}_{k,\varepsilon}) \\ &+ \sum_{j=0}^{k-1} \eta^* \left(\left(\prod_{i=1}^j \widehat{f}_{i,\varepsilon} \right) \cdot (\widehat{f}_{j+1} - \widehat{f}_{j+1,\varepsilon}) \cdot \left(\prod_{i=j+2}^k \widehat{f}_i \right) \right), \end{aligned}$$

and

$$\begin{aligned} (\mu^{*n} * \eta^{\otimes k})(f_1 \otimes \cdots \otimes f_k) &= (\mu^{*n} * \eta^{\otimes k})(f_{1,\varepsilon} \otimes \cdots \otimes f_{k,\varepsilon}) \\ &+ \sum_{j=0}^{k-1} (\mu^{*n} * \eta^{\otimes k}) \left(\left(\bigotimes_{i=1}^j f_{i,\varepsilon} \right) \otimes (f_{j+1} - f_{j+1,\varepsilon}) \otimes \left(\bigotimes_{i=j+2}^k f_i \right) \right), \end{aligned}$$

for all $n \geq 1$, with the convention that products or tensor products over empty sets are equal to one. By (3.1),

$$|\eta^*(\widehat{f}_{j+1} - \widehat{f}_{j+1,\varepsilon})| \leq \|f_{j+1} - f_{j+1,\varepsilon}\|_{L^1(\eta)} < \varepsilon,$$

for all $j = 1, \dots, k-1$, whence

$$|\eta^*(\widehat{f}_1 \cdots \widehat{f}_k) - \eta^*(\widehat{f}_{1,\varepsilon} \cdots \widehat{f}_{k,\varepsilon})| \leq \varepsilon \sum_{j=1}^k \prod_{i \neq j} \|f_{i,\varepsilon}\|_\infty.$$

Since η is μ -stationary, the marginal measures of $\mu^{*n} * \eta^{\otimes k}$ are all equal to η , and thus

$$|(\mu^{*n} * \eta^{\otimes k})(f_1 \otimes \cdots \otimes f_k) - (\mu^{*n} * \eta^{\otimes k})(f_{1,\varepsilon} \otimes \cdots \otimes f_{k,\varepsilon})| \leq \varepsilon \sum_{j=1}^k \prod_{i \neq j} \|f_{i,\varepsilon}\|_\infty,$$

for all $n \geq 1$. Since $\varepsilon > 0$ and n are arbitrary, and since

$$\lim_n (\mu^{*n} * \eta^{\otimes k})(f_{1,\varepsilon} \otimes \cdots \otimes f_{k,\varepsilon}) = \eta^*(\widehat{f}_{1,\varepsilon} \cdots \widehat{f}_{k,\varepsilon}),$$

(since $f_{1,\varepsilon}, \dots, f_{k,\varepsilon}$ are continuous), we conclude that

$$\lim_n (\mu^{*n} * \eta^{\otimes k})(f_1 \otimes \cdots \otimes f_k) = \eta^*(\widehat{f}_1 \cdots \widehat{f}_k).$$

□

3.2. Proximity and couplings

Let G be a lsc group and μ a probability measure on G . A μ -stationary probability measure η on M is μ -proximal if the conditional measure map $\omega \mapsto \eta_\omega$, restricted to a $\mu^{\mathbb{N}}$ -conull subset of Ω_η , takes values in the set of point measures on M .

Definition 3.9 (Couplings). Let $k \geq 2$ be an integer. A Borel probability measure κ on M^k is a k -coupling of (M, η) if the push-forward of κ to every M -factor equals η . We denote by η_{Δ_k} the k -coupling of (M, η) which is supported on the k -diagonal in M^k .

Remark 3.10. Every probability measure η on M has two "trivial" 2-couplings: the product measure $\eta \otimes \eta$ and the 2-diagonal measure η_{Δ_2} . If η is μ -stationary, then η_{Δ_2} is always μ -stationary, but $\eta \otimes \eta$ mostly fails to be μ -stationary (unless η is G -invariant).

Lemma 3.11. *Let μ be a probability measure on G , M a compact G -space and η a μ -stationary probability measure on M . The following conditions are equivalent.*

- (I) η is μ -proximal.
- (II) η_{Δ_2} is the only μ -stationary 2-coupling of (M, η) .
- (III) $\lim_n \mu^{*n} * \eta^{\otimes 4} = \eta_{\Delta_4}$.
- (IV) $\lim_n \mu^{*n} * \eta^{\otimes k} = \eta_{\Delta_k}$, for all $k \geq 2$.

Remark 3.12. While the equivalence between (I) and (II) is fairly standard (see e.g. [31] for related results), the equivalence between (I) and (III) seems to be new (at least we have not been able to find an explicit reference in the literature).

Proof. (I) \implies (II): Suppose that η is μ -proximal, and pick a μ -stationary self-coupling κ of the (G, μ) -space (M, η) . Then κ_ω is almost surely a self-coupling of η_ω , whence equal to $\eta_\omega \otimes \eta_\omega$ since η_ω is a point measure. Since

$$\eta = \int_{\Omega_\eta} \eta_\omega d\mu^{\otimes \mathbb{N}}(\omega) \quad \text{and} \quad \kappa = \int_{\Omega_\eta} \eta_\omega \otimes \eta_\omega d\mu^{\otimes \mathbb{N}}(\omega),$$

we conclude that $\kappa = \eta_{\Delta_2}$.

(II) \implies (IV): Fix $k \geq 2$, and note that

$$\kappa_k := \lim_n \mu^{*n} * \eta^{\otimes k} = \lim_n \int_{\Omega_\eta} (z_n(\omega)\eta)^{\otimes k} d\mu^{\otimes \mathbb{N}}(\omega) = \int_{\Omega_\eta} \eta_\omega^{\otimes k} d\mu^{\otimes \mathbb{N}}(\omega),$$

where $z_n(\omega) = \omega_1 \cdots \omega_n$ for $\omega \in \Omega_\eta$, is a μ -stationary k -coupling of (M, η) (the limit exists by Lemma 3.3). In particular, $\kappa_2 = \eta_{\Delta_2}$ by (II), and if $k > 2$, then the push-forward of κ_k to every $M \times M$ -factor must equal η_{Δ_2} . By induction, we conclude that $\kappa_k = \eta_{\Delta_k}$.

(IV) \implies (III): Trivial.

(III) \implies (I): To prove that η_ω is almost surely a point measure, it suffices to show that there is a \mathbb{P}_μ -conull subset $\Omega \subset G^{\mathbb{N}}$ such that

$$\eta_\omega(f_1 f_2) = \eta_\omega(f_1) \eta_\omega(f_2), \quad \text{for all } \omega \in \Omega \text{ and } f_1, f_2 \in C(M). \quad (3.2)$$

Since M is metrizable, $C(M)$ equipped with the uniform norm, is a separable Banach space. Let (f_i) be a countable norm-dense subset of $C(M)$. To prove (3.2), it is clearly enough to show that

$$\int_{G^{\mathbb{N}}} |\eta_\omega(f_i f_j) - \eta_\omega(f_i) \eta_\omega(f_j)|^2 d\mu^{\otimes \mathbb{N}}(\omega) = 0, \quad \text{for all } i, j.$$

Upon expanding the square, we see that this amounts to proving:

$$\kappa_2(f_i f_j \otimes f_i f_j) - 2\kappa_3(f_i f_j \otimes f_i \otimes f_j) + \kappa_4(f_i \otimes f_i \otimes f_j \otimes f_j) = 0, \quad (3.3)$$

for all i and j , where

$$\kappa_k = \int_{G^{\mathbb{N}}} \eta_\omega^{\otimes k} d\mu^{\otimes \mathbb{N}}(\omega), \quad \text{for } k = 2, 3, 4.$$

Since we assume that $\lim_n \mu^{*n} * \eta^{\otimes 4} = \eta_{\Delta_4}$, we have

$$\begin{aligned} \kappa_4 &= \int_{G^{\mathbb{N}}} \eta_{\omega}^{\otimes 4} d\mu^{\otimes \mathbb{N}}(\omega) \\ &= \lim_n \int_{G^{\mathbb{N}}} (z_n(\omega)_* \eta)^{\otimes 4} d\mu^{\otimes \mathbb{N}}(\omega) \\ &= \lim_n \mu^{*n} * \eta^{\otimes 4} = \eta_{\Delta_4}, \end{aligned}$$

and thus $\kappa_3 = \eta_{\Delta_3}$ and $\kappa_2 = \eta_{\Delta_2}$ as well. We conclude that

$$\kappa_2(f_i f_j \otimes f_i f_j) = \kappa_3(f_i f_j \otimes f_i \otimes f_j) = \kappa_4(f_i \otimes f_i \otimes f_j \otimes f_j) = \eta(f_i^2 f_j^2),$$

for all i and j , from which (3.3) readily follows, and we are done. \square

The following corollary is an immediate consequence of the equivalence between the conditions (I) and (III) in the lemma above.

Corollary 3.13. *Let μ_1 and μ_2 be probability measures on G , M a compact G -space and η a probability measure on M which is both μ_1 -stationary and μ_2 -stationary. Suppose that*

$$\mu_1^{*n} * \eta^{\otimes 4} = \mu_2^{*n} * \eta^{\otimes 4}, \quad \text{for all } n.$$

Then η is μ_1 -proximal if and only if it is μ_2 -proximal.

We stress that we did not assume in the previous corollary that the measures μ_1 and μ_2 are spread-out. However, in the next corollary, we shall assume that they are.

Corollary 3.14. *Let μ_1 and μ_2 be spread-out probability measures on G , M a compact G -space and η a probability measure on M which is both μ_1 -stationary and μ_2 -stationary. Suppose that $\mu_1 * \mu_2 = \mu_2 * \mu_1$. Then η is μ_1 -proximal if and only if it is μ_2 -proximal.*

Proof. We shall assume that η is μ_1 -proximal and prove that η_{Δ_2} is the only μ_2 -stationary self-coupling of (M, η) . In view of the equivalence between (I) and (II) in Lemma 3.11, this will finish the proof of the corollary. We fix a μ_2 -stationary self-joining κ of (M, η) . By Corollary 3.2 applied to the compact G -space $M \times M$, κ is also μ_1 -stationary, and thus equal to η_{Δ_2} since η is μ_1 -proximal. \square

4. PRELIMINARIES ON BOREL (G, μ) -SPACES

Let G be a lsc group, μ a probability measure on G , and (X, ξ) a Borel (G, μ) -space. We shall always assume that our spaces are *standard*. The space $\mathcal{H}^\infty(G, \mu)$ of bounded μ -harmonic functions on G is defined by

$$\mathcal{H}^\infty(G, \mu) = \{\varphi \in \mathcal{L}^\infty(G) : \varphi * \mu = \varphi\}.$$

If μ is spread-out, $\mathcal{H}^\infty(G, \mu) \subset C_b(G)$. We denote by $P_\xi : L^\infty(X, \xi) \rightarrow \mathcal{H}^\infty(G, \mu)$ the *Poisson transform* of (X, ξ) , which is defined by

$$P_\xi f(g) = \int_X f(gx) d\xi(x), \quad \text{for } g \in G \text{ and } f \in L^\infty(X, \xi).$$

If (X', ξ') is another Borel (G, μ) -space, and there are G -invariant conull subsets $X_o \subset X$ and $X'_o \subset X'$, and a Borel G -map $\pi : X_o \rightarrow X'_o$ such that the measure class of the push-forward measure $\pi_*(\xi|_{X_o})$ equals the measure class of ξ' , we say that π is a *measurable G -map*. If moreover, $\pi_*(\xi|_{X_o}) = \xi'_{X'_o}$, we say that π is *measure-preserving*. To make notation less heavy, we shall suppress the dependences on the subsets X_o and X'_o , and simply write $\pi : (X, \xi) \rightarrow (X', \xi')$ if π is a measurable G -map, and then emphasize if this map is measure-preserving or not. If π is measure-preserving, we also say that (X', ξ') is

a G -factor of (X, ξ) , and if π admits a measurable inverse, which is a measure-preserving measurable G -map, we say that the Borel (G, μ) -spaces (X, ξ) and (X', ξ') are *measurably G -isomorphic*.

4.1. Compact models

Definition 4.1 (Compact model). Let (X, ξ) be a Borel (G, μ) -space, M a compact G -space and η a μ -stationary probability measure on M . If (M, η) is measurably G -isomorphic to (X, ξ) , we say that (M, η) is a *compact model* of (X, ξ) .

Proposition 4.2. *Let (X, ξ) be a Borel (G, μ) -space.*

- (I) *There exists a compact model for (X, ξ) .*
- (II) *If (X', ξ') is a Borel (G, μ) -space, and $\pi : (X, \mu) \rightarrow (X', \xi')$ is a measure-preserving G -map, and (M, η) and (M', η') are compact models for (X, ξ) and (X', ξ') respectively, then there is a Borel G -map $\sigma : M \rightarrow M'$, which intertwines π , such that*

$$\sigma_*\eta = \eta' \quad \text{and} \quad \sigma_*\eta_\omega = \eta'_\omega, \quad \text{for } \mu^{\otimes \mathbb{N}}\text{-almost every } \omega,$$

where $\omega \mapsto \eta_\omega$ and $\omega \mapsto \eta'_\omega$ denote the conditional measure maps associated with η and η' respectively.

Proof. (I) and the first part of (II) are stated, with appropriate references, in [7, Theorem 2.1], while the second part of (II) is proved in [7, Corollary 2.7]. \square

The key point of Proposition 4.2 is that every Borel (G, μ) -space can be endowed with a (measure class) of conditional measure maps. Indeed, if (M, η) is a compact model for this space, then there is a conditional measure map by Lemma 3.3, and if (M', η') is another compact model, then the induced measure-preserving G -map between (M, η) and (M', η') (which is Borel), pushes the conditional measure map for (M, η) to the conditional measure map for (M', η') (modulo $\mu^{\mathbb{N}}$ -null sets) by (II) in Proposition 4.2.

4.2. Proximal Borel (G, μ) -spaces

Definition 4.3 (μ -boundary). We say that a Borel (G, μ) -space (X, ξ) is *μ -proximal* (or is a *μ -boundary*) if every compact model of (X, ξ) is μ -proximal.

Remark 4.4. By Proposition 4.2, between any two different compact models of (X, ξ) , there is a measure-preserving Borel G -isomorphism, which maps the conditional measures to conditional measures (modulo $\mu^{\mathbb{N}}$ -null sets), so in particular the notion of μ -proximality for Borel (G, μ) -spaces is well-defined (and restricts to μ -proximality for compact G -spaces).

By passing to compact models in Lemma 3.11, Corollary 3.13 and Corollary 3.14, we can now immediately formulate the following results.

Proposition 4.5. *Let G be a lsc group and μ a probability measure on G . Let (X, ξ) be a μ -proximal Borel (G, μ) -space. Then, for every $k \geq 2$ and for every k -tuple (f_1, \dots, f_k) of bounded Borel functions on X ,*

$$\lim_n (\mu^{*n} * \xi^{\otimes k})(f_1 \otimes \dots \otimes f_k) = \xi(f_1 \dots f_k).$$

Remark 4.6. A word of caution here: we do not claim that upon passing to a compact model of (X, ξ) , the functions f_1, \dots, f_k become continuous, and Lemma 3.11 can be applied of-the-shelf. Rather, if (M, η) is a τ -proximal compact model of (X, ξ) , we apply (IV) in Lemma 3.11 to conclude that $\lim_n \mu^{*n} * \eta^{\otimes k} = \eta_{\Delta_k}$ in the weak*-sense. Then, just

as in the proof of Lemma 3.8, we conclude that the convergence also holds for arbitrary k -tuples of bounded Borel functions on M .

Proposition 4.7. *Let G be a lsc group, let μ_1 and μ_2 be probability measures on G and let (X, ξ) be a Borel (G, μ_1) -space which is also a Borel (G, μ_2) -space.*

- (I) *If $\mu_1^{*n} * \xi^{\otimes 4} = \mu_2^{*n} * \xi^{\otimes 4}$ for all n , then ξ is μ_1 -proximal if and only if it is μ_2 -proximal.*
- (II) *If μ_1 and μ_2 are spread-out and $\mu_1 * \mu_2 = \mu_2 * \mu_1$, then ξ is μ_1 -proximal if and only if it is μ_2 -proximal.*

Let us also record here some following well-known properties of the canonical quasi-factor, introduced in Subsection 3.1 above.

Proposition 4.8. *Let M be a compact G -space, let η be a μ -stationary probability measure on M and let $(\text{Prob}(M), \eta^*)$ denote the canonical quasi-factor associated with (M, η) .*

- (I) *$(\text{Prob}(M), \eta^*)$ is a μ -boundary.*
- (II) *If η is μ -proximal, then $(\text{Prob}(M), \eta^*)$ is measurably G -isomorphic to (M, η) .*

Proof. (I) can for instance be found in [31, Proposition 3.2], while (II) is contained in [33, Theorem 4.3]. \square

4.3. The Poisson boundary of a measured group

The following fundamental construction is due to Furstenberg [29].

Theorem 4.9. *For every measured group (G, μ) there exists a μ -proximal Borel (G, μ) -space (B, ν) (which is unique modulo measurable G -isomorphisms) with the following properties:*

MAXIMALITY: *Every μ -boundary is a G -factor of (B, ν) .*

POISSON REPRESENTATION: *The Poisson transform $P_\nu : L^\infty(B, \nu) \rightarrow \mathcal{H}^\infty(G, \mu)$ is an isometric isomorphism.*

Remark 4.10. We stress that we assume in Theorem 4.9 that μ is spread-out.

Definition 4.11 (Poisson boundary). The Borel (G, μ) -space (B, ν) (or any of its measurable G -isomorphic images) in Theorem 4.9 is called the *Poisson boundary* of the measured group (G, μ) .

4.4. Dense subgroups

We shall now prove one of the key ingredient in the proof of (II) in Theorem 1.9. Although we have not seen this result explicitly stated in the literature, it is a fairly immediate consequence of the fundamental works of Mackey and Ramsey, as outlined in the appendix to Zimmer's book [63]. For completeness, we shall provide the details. We use the same terminology of measurable G -maps and G -factors as in the beginning of the section, but with the difference that our measures are not necessarily μ -stationary (but only quasi-invariant under G , and certain dense subgroups thereof). For this reason we shall refer to Borel G -spaces, and not Borel (G, μ) -spaces, and we shall not insist that our maps are measure-preserving, but only that they map measure classes to measure classes. As before, we always assume that the Borel spaces involved are standard.

In what follows, let

- G be a lcsc group and G_o a dense subgroup of G .
- (X, ξ) be a Borel G -space and (Y, κ) a Borel G_o -space.
- $\alpha : (X, \xi) \rightarrow (Y, \nu)$ a measurable G_o -map.

Proposition 4.12. *There exist*

- a Borel G -space (Z, λ) ,
- a measurable G_o -isomorphism $\beta : (Y, \kappa) \rightarrow (Z, \lambda)$,
- a measurable G -map $\gamma : (X, \xi) \rightarrow (Z, \lambda)$,

such that

$$\begin{array}{ccc} (X, \xi) & & \\ \alpha \downarrow & \searrow \gamma & \\ (Y, \kappa) & \xrightarrow{\beta} & (Z, \lambda) \end{array}$$

In other words, every G_o -factor of the Borel G_o -space (X, ξ) is measurably G_o -isomorphic to a G -factor of the Borel G -space (X, ξ) , viewed as a G_o -space.

Proof. The proof consists of two main steps, which are both composed of references to the appendix of Zimmer's book [63].

STEP I: CONSTRUCTION OF (Z, λ) AND β .

Let (W, ϑ) be a Borel G -space. Since G is lcsc, G acts continuously on the Banach space $L^1(W, \vartheta)$ and thus on $L^\infty(W, \vartheta)$ endowed with the weak*-topology (the G -action is not necessarily continuous with respect to the norm-topology). We write

$$B(W, \vartheta) = \{f \in L^\infty(W, \vartheta) : f^2 = f\},$$

which is clearly a G -invariant weak*-compact set. If (W', ϑ') is another Borel G -space, and $\delta : (W, \vartheta) \rightarrow (W', \vartheta')$ is a measurable G -map, then $\delta^* : B(W', \vartheta') \rightarrow B(W, \vartheta)$ is a continuous, G -equivariant and injective map. In particular, $\delta^*(B(W', \vartheta'))$ is a weak*-compact and G -invariant subset of $B(W, \vartheta)$.

Let us now specialize this to our setting. Since G_o is dense in G and G acts continuously on $B(X, \xi)$ and since $\alpha^*(B(Y, \kappa)) \subset B(X, \xi)$ is G_o -invariant and weak*-compact, we conclude that $\alpha^*(B(Y, \kappa))$ is G -invariant. Since α^* is injective, we can thus endow $B(Y, \kappa)$ with a jointly continuous G -action so that α^* is G -equivariant, by setting

$$g.f := (\alpha^*)^{-1}(g.\alpha^*(f)), \quad \text{for } g \in G \text{ and } f \in B(Y, \kappa).$$

By [63, Theorem B.10], there exist a Borel G -space (Z, λ) and a G -equivariant Boolean map $\phi : B(Z, \lambda) \rightarrow B(Y, \kappa)$. Since ϕ is also G_o -equivariant, [63, Corollary B.7] asserts that there exists a measurable G_o -isomorphism $\beta : (Y, \kappa) \rightarrow (Z, \lambda)$, such that $\phi = \beta^*$.

STEP II: CONSTRUCTION OF γ

Note that $\psi : \alpha^* \circ \beta^* : B(Z, \lambda) \rightarrow B(X, \xi)$ is a G -equivariant Boolean G -map, so by [63, Corollary B.6], there exists a measurable G -map $\gamma : (X, \xi) \rightarrow (Z, \lambda)$ such that $\gamma^* = \psi$, and thus $\gamma = \beta \circ \alpha$. \square

We stress that the maps β and γ in Proposition 4.12 are not measure-preserving in general. However, the following observation tells us that in the setting of *ergodic* Borel (G, μ) -spaces, for μ spread-out, these maps are automatically measure-preserving.

Lemma 4.13. *Let μ be a spread-out probability measure on G and let (X, ξ) and (X', ξ') be ergodic Borel (G, μ) -spaces. Then every measurable G -map*

$$\pi : (X, \xi) \rightarrow (X', \xi')$$

is measure-preserving.

Proof. By Proposition 4.2 we may without loss of generality assume that X and X' are compact G -spaces, and that π is Borel. Since π is a G -equivariant and ξ is ergodic, $\pi_*\xi$ is a μ -stationary and ergodic probability measure on X' which is absolutely continuous with respect to ξ' , whence equal to ξ' by (III) in Lemma 3.1. \square

If we combine this lemma with Proposition 4.12, we arrive at the following corollary.

Corollary 4.14. *Let G be a lcsc group and G_o a dense subgroup of G . Suppose that μ and μ_o are spread-out probability measures on G and G_o respectively, and let (X, ξ) be an ergodic Borel (G, μ) -space which is also a Borel (G_o, μ_o) -space. Then every G_o -factor of (X, ξ) is measurably G_o -isomorphic (in a measure-preserving way) to a G -factor of (X, ξ) .*

4.5. Transitive G -spaces

Let G be a lcsc group and P a closed subgroup of G . By [61, Section V.4], the quotient space G/P admits a unique G -invariant measure class (but not necessarily a G -invariant measure). By [50, Section IV.2, Proposition 2.4b)], if (X, ξ) is a Borel G -space and π is measurable G -equivariant map from G/P to X , which maps the G -invariant measure class on G/P to the measure class of ξ , then (X, ξ) is measurably G -isomorphic to a quotient space of the form G/Q , where Q is a *closed subgroup* of G which contains P (here, G/Q is endowed with the unique G -invariant measure class). From this, and Corollary 4.14 above, we can now conclude the following result.

Corollary 4.15. *Let G_o be a dense subgroup of G , μ_o a spread-out probability measure on G_o and ξ a μ_o -stationary probability measure on G/P which is absolutely continuous with respect to the unique G -invariant measure class. Then every G_o -factor of the Borel (G_o, μ_o) -space $(G/P, \xi)$ is G_o -isomorphic to $(G/Q, \xi')$, where Q is a closed subgroup of G which contains P , and ξ' is μ_o -stationary. In particular, if P is a maximal proper closed subgroup of G , then the Borel (G_o, μ_o) -space $(G/P, \xi)$ is prime.*

4.6. The Furstenberg entropy of Borel (G, μ) -spaces

Let G be a lcsc group and let μ be a Borel probability measure on G (not necessarily spread-out). Suppose that (X, ξ) is a Borel space endowed with a jointly measurable action of G , preserving the measure class of ξ . We assume that ξ is μ -stationary. The *Furstenberg entropy* $h_\mu(X, \xi)$ is defined by

$$h_\mu(X, \xi) = \int_G \left(\int_X -\log \frac{dg^{-1}\xi}{d\xi}(x) d\xi(x) \right) d\mu(g),$$

whenever the integral exists and is finite. Note that $h_{\delta_e}(X, \xi) = 0$. If μ is spread-out, we define the *entropy* $h(\mu)$ as the Furstenberg entropy of the Poisson boundary of the measured group (G, μ) . We define

$$\mathcal{A}_\xi = \{ \mu' \in \text{Prob}(G) \mid \mu' * \xi = \xi, \ h_{\mu'}(X, \xi) \text{ exists and is finite} \}.$$

We leave the proof of the following easy lemma to the reader.

Lemma 4.16. *The set $\mathcal{A}_\xi \subset \text{Prob}(G)$ is convex and closed under convolution. Furthermore, the map $\mu' \mapsto h_{\mu'}(X, \xi)$ is affine on the \mathcal{A}_ξ , and satisfies*

$$h_{\mu' * \mu''}(X, \xi) = h_{\mu'}(X, \xi) + h_{\mu''}(X, \xi), \quad \text{for all } \mu', \mu'' \in \mathcal{A}_\xi.$$

5. PROOF OF THEOREM 1.6

Throughout this section, let Γ denote a countable discrete group, H a totally disconnected lsc group and $\rho : \Gamma \rightarrow H$ a homomorphism with dense image. We shall fix a compact open subgroup L of H and set $\Lambda = \rho^{-1}(L) < \Gamma$. Then (Γ, Λ) is a Hecke pair, and the map

$$\bar{\rho} : \Gamma/\Lambda \rightarrow H/L, \quad \gamma\Lambda \mapsto \rho(\gamma)L$$

is a bijection. We write α for the canonical quotient map from Γ to Γ/Λ , and if $\varphi \in C_b(H)$, we define $\varphi_L \in C_b(H/L)$ by

$$\varphi_L(hL) = \int_L \varphi(hl) dm_L(l), \quad \text{for } hL \in H/L,$$

where m_L denotes the Haar probability measure on L . We define the *affine* map

$$\text{Prob}(\Gamma; \Lambda) \rightarrow \text{Prob}(H, L), \quad \tau \mapsto \theta_\tau$$

by

$$\theta_\tau(\varphi) = (\alpha_*\tau)(\varphi_L \circ \bar{\rho}), \quad \text{for } \varphi \in C_b(H). \quad (5.1)$$

By construction, θ_τ is right- L -invariant. To prove that it is also left- L -invariant, and thus belongs to $\text{Prob}(H, L)$, pick $l \in \rho(\Lambda)$ and $\lambda_l \in \Lambda$ such that $\rho(\lambda_l) = l$. Since τ is Λ -absorbing, $\tau_*\alpha$ is left- Λ -invariant, and thus

$$\theta_\tau(\varphi(l \cdot)) = \alpha_*\tau((\varphi_L \circ \bar{\rho})(\lambda_l \cdot)) = \alpha_*\tau((\varphi_L \circ \bar{\rho})(\cdot)) = \theta_\tau(\varphi),$$

for all $\varphi \in C_b(H)$, whence θ_τ is left $\rho(\Lambda)$ -invariant (and thus left L -invariant, since $\rho(\Lambda)$ is dense in L , and H -stabilizers of probability measures on H are *closed* subgroups).

It will be convenient to use the following notation: if τ is a Λ -absorbing probability measure on Γ , set

$$\bar{\tau} = \alpha_*\tau \quad \text{and} \quad \bar{\theta}_\tau = (\bar{\rho})_*\bar{\tau},$$

so that $\bar{\theta}_\tau(\varphi_L) = \theta_\tau(\varphi)$ for all $\varphi \in C_b(H)$. In particular,

$$\bar{\tau}(\varphi_L \circ \bar{\rho}) = \theta_\tau(\varphi), \quad \text{for all } \varphi \in C(H) \cap \mathcal{L}^1(H, \theta_\tau). \quad (5.2)$$

Since $\bar{\rho}$ is a bijection and $\tau \mapsto \bar{\tau}$ is surjective, we conclude that $\tau \mapsto \theta_\tau$ is surjective.

Proof that the map is a monoid homomorphism

We recall from (2.2) that

$$\alpha_*(\tau_1 * \tau_2)(\gamma\Lambda) = \sum_{\eta\Lambda} \bar{\tau}_1(\eta\Lambda) \bar{\tau}_2(\eta^{-1}\gamma\Lambda), \quad \text{for } \gamma\Lambda \in \Gamma/\Lambda.$$

Hence, for all $\varphi \in C_b(H)$,

$$\begin{aligned}
\theta_{\tau_1 * \tau_2}(\varphi) &= \alpha_*(\tau_1 * \tau_2)(\varphi_L \circ \bar{\rho}) = \sum_{\gamma \Lambda} \sum_{\eta \Lambda} \varphi_L(\bar{\rho}(\gamma \Lambda)) \bar{\tau}_1(\eta \Lambda) \bar{\tau}_2(\eta^{-1} \gamma \Lambda) \\
&= \sum_{\eta \Lambda} \left(\sum_{\gamma \Lambda} \varphi_L(\rho(\eta) \bar{\rho}(\gamma \Lambda)) \bar{\tau}_2(\gamma \Lambda) \right) \bar{\tau}_1(\eta \Lambda) \\
&= \sum_{\eta \Lambda} \bar{\theta}_{\tau_2}(\varphi_L(\rho(\eta) \cdot)) \bar{\tau}_1(\eta \Lambda) = \sum_{\eta \Lambda} \bar{\theta}_{\tau_2}((\varphi(\rho(\eta) \cdot))_L) \bar{\tau}_1(\eta \Lambda) \\
&= \int_{H/L} \bar{\theta}_{\tau_2}((\varphi(h \cdot))_L) d\bar{\theta}_{\tau_1}(hL) = (\theta_{\tau_1} * \theta_{\tau_2})(\varphi),
\end{aligned}$$

where we in the fourth identity have used (5.2) and in the fifth identity the fact that the assignment $\varphi \mapsto \varphi_L$ is left- H -equivariant, i.e. $\varphi_L(h \cdot) = (\varphi(h \cdot))_L$ for all $h \in H$. Since τ_1 and τ_2 are arbitrary in $\text{Prob}(\Gamma; \Lambda)$, we conclude that

$$\theta_{\tau_1 * \tau_2} = \theta_{\tau_1} * \theta_{\tau_2}, \quad \text{for all } \tau_1, \tau_2 \in \text{Prob}(\Gamma; \Lambda).$$

Finally, $\theta_{\delta_e} = m_L$ and thus the map $\tau \mapsto \theta_\tau$ is a monoid homomorphism.

Proof of Property (P1)

Suppose that $\varphi \in C(H) \cap \mathcal{L}^1(H, \theta_\tau)$ is right L -invariant. Then,

$$\varphi_L \circ \bar{\rho} \circ \alpha = \varphi \circ \rho,$$

whence $\theta_\tau(\varphi) = \rho_* \tau(\varphi)$ by (5.2).

Proof of Property (P2)

Let us fix $\tau \in \text{Prob}(\Gamma; \Lambda)$. Since θ_τ is bi- L -invariant, its support must be bi- L -invariant, and since $\rho(\Gamma)$ is dense in H , it suffices to prove the inclusions

$$\rho(\text{supp } \tau) \subset \text{supp } \theta_\tau \cap \rho(\Gamma) \subset \rho(\text{supp }(\tau))L,$$

or equivalently, that

$$\rho(\gamma) \in \text{supp } \theta_\tau \iff \rho(\gamma)L \subset \rho(\text{supp } \tau)L.$$

To show this, pick $\gamma \in \Gamma$ and an open identity neighborhood in H , which we may assume is contained in L . If we unwrap the definition of the measure θ_τ (see (5.1)), we get

$$\begin{aligned}
\theta_\tau(\rho(\gamma)U) &= \int_{H/L} \left(\int_L \chi_{\rho(\gamma)U}(hl) dm_L(l) \right) d\bar{\theta}_\tau(hL) \\
&= \sum_{\eta \Lambda} \left(\int_L \chi_{\rho(\gamma)U}(\rho(\eta)l) dm_L(l) \right) \bar{\tau}(\eta \Lambda) \\
&= \sum_{\eta \Lambda} \sum_{\lambda \in \Lambda} \left(\int_L \chi_{\rho(\gamma)U}(\rho(\eta)l) dm_L(l) \right) \tau(\eta \lambda)
\end{aligned}$$

and thus $\theta_\tau(\rho(\gamma)U) > 0$ if and only if there exist $\eta \in \Gamma$ and $\lambda \in \Lambda$ such that

$$m_L(\rho(\eta^{-1} \gamma)U \cap L) > 0 \quad \text{and} \quad \tau(\eta \lambda) > 0.$$

Since U is open, the first condition is equivalent to saying that $\rho(\eta^{-1}\gamma)U \cap L \neq \emptyset$. Since the second condition is left- Λ -invariant, and U is an arbitrary clopen identity neighborhood in $L \subset H$, we conclude that

$$\rho(\gamma) \in \text{supp } \theta_\tau \iff \rho(\gamma)L \subset \rho(\text{supp } \tau)L,$$

which finishes the proof.

Proof of Property (P3)

Let X be a Borel H -space, ξ a L -invariant Borel measure on X and τ a Λ -absorbing probability measure on Γ . For every bounded Borel function f on X , the function

$$\varphi_f(h) = h\xi(f), \quad h \in H$$

is right L -invariant, and thus continuous since H is open, whence by (P1) above,

$$(\theta_\tau * \xi)(f) = \int_H \varphi_f(h) d\theta_\tau(h) = \sum_{\gamma \in \Gamma} \tau(\gamma) \rho(\gamma)\xi(f) = (\tau * \xi)(f).$$

Since f is arbitrary, we conclude that $\theta_\tau * \xi = \tau * \xi$.

Proof of Property (P4)

Let τ be a Λ -absorbing probability measure on Γ and fix a Borel (H, θ_τ) -space (X, ξ) . Since ξ is L -invariant, so is $\xi^{\otimes 4}$ under the diagonal H -action on X^4 . Using that $\tau \mapsto \theta_\tau$ is a monoid homomorphism and property (P3) above, we see that

$$\tau^{*n} * \xi^{\otimes 4} = \theta_{\tau^{*n}} * \xi^{\otimes 4} = \theta_\tau^{*n} * \xi^{\otimes 4}, \quad \text{for all } n \geq 1,$$

and thus ξ is τ -proximal if and only if it is θ_τ -proximal by (i) in Proposition 4.7.

Remark 5.1. Let us highlight a hidden subtlety in the proof of (P4). In view of our Lemma 3.11 above, the key point is really to show that if (M, η) is a compact model of a θ_τ -proximal Borel (H, θ_τ) -space, then η_{Δ_2} is the only τ -stationary 2-coupling of (M, η) . However, a compact H -space could in general admit τ -stationary probability measures, which are not θ_τ -stationary. To circumvent this issue, we found it necessary to establish Proposition 4.7. We stress that we are not aware of an argument which would prove (P4) for an *arbitrary* Furstenberg discretization of θ_τ .

6. PROOF OF COROLLARY 1.9

Let (Γ, Λ) be a Hecke pair and (H, L, ρ) a Hecke completion of (Γ, Λ) . Fix a spread-out and Λ -absorbing probability measure τ on Γ , and write (H, θ_τ) for the Hecke completion of (Γ, τ) with respect to ρ . Let (B_τ, ν_τ) and $(B_{\theta_\tau}, \nu_{\theta_\tau})$ denote the Poisson boundaries of the measured groups (Γ, τ) and (H, θ_τ) respectively.

Proof of (I)

Since ν_τ is θ_τ -proximal, it is also τ -proximal by (P4) in Theorem 1.6, and thus $(B_{\theta_\tau}, \nu_{\theta_\tau})$ is a τ -boundary.

Proof of (II)

Let M be a compact Γ -space and η a τ -stationary probability measure on M . Suppose that η is Λ -invariant.

Lemma 6.1. *There is a Γ -equivariant, linear and unital bounded map*

$$T_\eta : C(M) \rightarrow L^\infty(B_{\theta_\tau}, \nu_{\theta_\tau})$$

with the property that

$$\gamma\eta(f) = \rho(\gamma)\nu_{\theta_\tau}(T_\eta f), \quad \text{for all } \gamma \in \Gamma \text{ and } f \in C(M).$$

Proof. Since η is Λ -invariant, for every $f \in C(M)$, the Poisson transform $P_\eta f$ is a bounded, τ -harmonic and right Λ -invariant function on Γ . Since

$$\bar{\rho} : \Gamma/\Lambda \rightarrow H/L, \quad \gamma\Lambda \mapsto \rho(\gamma)L,$$

is a bijection, $\varphi_f(h) := P_\eta f(\bar{\rho}^{-1}(hL))$ is a well-defined bounded and right L -invariant (hence continuous) function on H , which satisfies $\varphi_f \circ \rho = P_\eta f$. By Property (P1) in Theorem 1.6, applied to left-translates of φ_f , we conclude that φ_f is θ_τ -harmonic. By Theorem 4.9, the Poisson transform $P_{\nu_{\theta_\tau}}$ sets up an isometric bijection between $L^\infty(B_{\theta_\tau}, \nu_{\theta_\tau})$ and $\mathcal{H}^\infty(H, \theta_\tau)$, so we can define $T_\eta : C(M) \rightarrow L^\infty(B_{\theta_\tau}, \nu_{\theta_\tau})$ by

$$T_\eta f = P_{\nu_{\theta_\tau}}^{-1} \varphi_f, \quad \text{for } f \in C(M),$$

which is clearly Γ -equivariant and bounded, and satisfies $\gamma\eta(f) = \rho(\gamma)\nu_{\theta_\tau}(T_\eta f)$ for all $\gamma \in \Gamma$ and $f \in C(M)$. \square

We recall the following notation from Subsection 3.1: if $f \in C(M)$ and $\kappa \in \text{Prob}(M)$, we define $\widehat{f}(\kappa) = \kappa(f)$. If we endow $\text{Prob}(M)$ with the weak*-topology, \widehat{f} is a continuous function on $\text{Prob}(M)$.

Corollary 6.2. *There is a measure-preserving Γ -map $\pi_\eta : (B_{\theta_\tau}, \nu_\theta) \rightarrow (\text{Prob}(M), (\pi_\eta)_* \nu_\theta)$ such that*

$$\widehat{f} \circ \pi_\eta = T_\eta f, \quad \text{for all } f \in C(M),$$

in the L^∞ -sense.

Proof. Since M is compact and metrizable, $C(M)$ (equipped with the uniform norm) is separable Banach space. Fix a countable dense subset (f_i) in $C(M)$. Since Γ is countable, we may without loss of generality assume that this set is closed under composition with elements from Γ . Let us choose, for every i , a Borel function $\widetilde{T}_\eta f_i$ on B_{θ_τ} which represents the function class $T_\eta f_i$ in $L^\infty(B_{\theta_\tau}, \nu_{\theta_\tau})$. Since Γ is countable, we can find a Γ -invariant ν_{θ_τ} -conull subset $B' \subset B_{\theta_\tau}$ such that for every i and $\gamma \in \Gamma$,

$$T_\eta(\widetilde{f_i \circ \gamma})(b) = \widetilde{T}_\eta f_i(\gamma b), \quad \text{for all } b \in B'. \quad (6.1)$$

One readily checks that the functionals on the \mathbb{Q} -linear span of (f_i) , defined by

$$\pi_\eta(b)(f_i) := \widetilde{T}_\eta f_i(b), \quad \text{for } b \in B',$$

for every i , are bounded, positive and unital, and thus extend by a straightforward approximation argument, to probability measures on M (via Riesz representation theorem). By dominated convergence and (6.1), π_η is a Γ -equivariant map from B' to $\text{Prob}(M)$, which satisfies $\widehat{f} \circ \pi_\eta = T_\eta f$ for all $f \in C(M)$ (in the L^∞ -sense) by construction. \square

It remains to prove that $(\pi_\eta)_*\nu_{\theta_\tau} = \eta^*$, where η^* denote the canonical quasi-factor associated with (M, η) , as introduced in Subsection 3.1. By the discussion in this subsection, this amounts to showing

$$\eta^*(\widehat{f}_1 \cdots \widehat{f}_k) = (\pi_\eta)_*\nu_{\theta_\tau}(\widehat{f}_1 \cdots \widehat{f}_k),$$

for all $f_1, \dots, f_k \in C(M)$ and $k \geq 1$. By Corollary 6.2, we have

$$(\pi_\eta)_*\nu_{\theta_\tau}(\widehat{f}_1 \cdots \widehat{f}_k) = \nu_{\theta_\tau}(\widehat{f}_1 \circ \pi_\eta \cdots \widehat{f}_k \circ \pi_\eta) = \nu_{\theta_\tau}(T_\eta f_1 \cdots T_\eta f_k),$$

for all $f_1, \dots, f_k \in C(M)$ and $k \geq 1$, so it suffices to show that

$$\eta^*(\widehat{f}_1 \cdots \widehat{f}_k) = \nu_{\theta_\tau}(T_\eta f_1 \cdots T_\eta f_k).$$

Let us from now on fix k and a k -tuple f_1, \dots, f_k in $C(M)$. By Lemma 3.8,

$$\eta^*(\widehat{f}_1 \cdots \widehat{f}_k) = \lim_{n \rightarrow \infty} (\tau^{*n} * \eta^{\otimes k})(f_1 \otimes \cdots \otimes f_k).$$

Using Lemma 6.1, we see that

$$\begin{aligned} (\tau^{*n} * \eta^{\otimes k})(f_1 \otimes \cdots \otimes f_k) &= \sum_{\gamma \in \Gamma} \gamma \eta(f_1) \cdots \gamma \eta(f_k) \tau^{*n}(\gamma) \\ &= \sum_{\gamma \in \Gamma} \rho(\gamma) \nu_{\theta_\tau}(T_\eta f_1) \cdots \rho(\gamma) \nu_{\theta_\tau}(T_\eta f_k) \tau^{*n}(\gamma), \end{aligned}$$

for all n . Since θ_τ is L -invariant, $h \mapsto h\nu_{\theta_\tau}(f_j)$ is right L -invariant for every $j = 1, \dots, k$, and thus by Property (P1) in Theorem 1.6, we have

$$\begin{aligned} \sum_{\gamma \in \Gamma} \rho(\gamma) \nu_{\theta_\tau}(T_\eta f_1) \cdots \rho(\gamma) \nu_{\theta_\tau}(T_\eta f_k) \tau^{*n}(\gamma) &= \int_H h \nu_{\theta_\tau}(T_\eta f_1) \cdots h \nu_{\theta_\tau}(T_\eta f_k) d\theta_\tau^{*n}(h) \\ &= (\theta_\tau^{*n} * \nu_{\theta_\tau}^{\otimes k})(T_\eta f_1 \otimes \cdots \otimes T_\eta f_k), \end{aligned}$$

for all n . Since $(B_{\theta_\tau}, \nu_{\theta_\tau})$ is θ_τ -proximal, Proposition 4.5 tells us that

$$\lim_n (\theta_\tau^{*n} * \nu_{\theta_\tau}^{\otimes k})(T_\eta f_1 \otimes \cdots \otimes T_\eta f_k) = \nu_{\theta_\tau}(T_\eta f_1 \cdots T_\eta f_k),$$

and thus

$$\lim_n (\tau^{*n} * \eta^{\otimes k})(f_1 \otimes \cdots \otimes f_k) = \nu_{\theta_\tau}(T_\eta f_1 \cdots T_\eta f_k).$$

From the discussion above, this proves that $(\pi_\eta)_*\nu_{\theta_\tau} = \eta^*$.

Finally, if (M, η) is τ -proximal, then Proposition 4.8 (II) shows that $(\text{Prob}(M), \eta^*)$ is Γ -isomorphic to (M, η) , whence the argument above shows that (M, η) is a θ_τ -boundary.

Proof of (III)

The Γ -action on (B_τ, ν_τ) is amenable by [62, Theorem 5.2], and thus the Λ -action is amenable as well by [63, Theorem 4.3.5]. If (B_τ, ν_τ) is measurably Γ -isomorphic to $(B_{\theta_\tau}, \nu_{\theta_\tau})$, then, since ν_{θ_τ} is L -invariant and $\Lambda = \rho^{-1}(L)$, we see that ν_τ is Λ -invariant. By [63, Proposition 4.3.3], this forces Λ to be an amenable group.

Proof of (IV)

Suppose that Λ is an amenable group and that (B_τ, ν_τ) admits a compact model (M, η) which is uniquely τ -stationary. Since τ is Λ -absorbing and Λ is amenable, (II) in Theorem 1.4 applied to the weak*-compact set $C = \text{Prob}(M)$, asserts that there is at least one τ -stationary and Λ -invariant probability measure on M . By unique τ -stationarity, we conclude that η is Λ -invariant. Since η is τ -proximal, (II) above implies that (M, η) is a θ_τ -boundary. Furthermore, by (I) above, $(B_{\theta_\tau}, \nu_{\theta_\tau})$ is a τ -boundary. We conclude that there are measure-preserving Γ -maps

$$(B_\tau, \nu_\tau) \xrightarrow{\pi_1} (B_{\theta_\tau}, \nu_{\theta_\tau}) \xrightarrow{\pi_2} (B_\tau, \nu_\tau).$$

In particular, $\pi := \pi_2 \circ \pi_1 : (B_\tau, \nu_\tau) \rightarrow (B_\tau, \nu_\tau)$ is a measure-preserving Γ -map, whence, by [33, Proposition 3.2], almost everywhere equal to the identity map, and thus (B_τ, ν_τ) and $(B_{\theta_\tau}, \nu_{\theta_\tau})$ are measurably Γ -isomorphic.

7. PROOF OF COROLLARY 1.10

Let $p \geq 2$ be a prime number. By [14, Corollary 1.4], there is a free group Γ of finite rank $r = r(p)$ and an *injective* homomorphism $\rho : \Gamma \rightarrow \text{SL}_2(\mathbb{Q}_p)$ with dense image (one can also appeal to Ihara's Theorem [39] which in particular implies that every torsion-free subgroup of $\text{SL}_2(\mathbb{Q}_p)$ is free). Set

$$H = \text{SL}_2(\mathbb{Q}_p) \quad \text{and} \quad L = \text{SL}_2(\mathbb{Z}_p) \quad \text{and} \quad \Lambda = \rho^{-1}(L).$$

Then (Γ, Λ) is a Hecke pair and (H, L, ρ) is a completion triple of (Γ, Λ) . Let P denote the closed solvable (hence amenable) subgroup of H consisting of upper triangular matrices. Since $H = LP$, there is a unique L -invariant Borel probability measure ν_o on H/P . Let τ be a spread-out and finitely supported Λ -absorbing probability measure on Γ . Such a measure exists by (III) in Theorem 1.4. By (P2) in Theorem 1.6, θ_τ is spread-out and compactly supported, and by [35, Theorem 15.5], the quotient space $(H/P, \nu_o)$ is the Poisson boundary of (H, θ_τ) . By (I) in Corollary 1.4, $(H/P, \nu_o)$ is a τ -boundary. We note that

$$\text{Stab}_\Gamma(hP) = \rho^{-1}(\Gamma \cap hPh^{-1}), \quad \text{for all } hP \in H/P.$$

In particular, $\text{Stab}_\Gamma(hP)$ is an amenable subgroup of Γ for every $hP \in H/P$, since ρ is injective and P is solvable. Since a free group of finite rank only has countably many amenable subgroups (the trivial group and cyclic subgroups), the map $hP \mapsto \text{Stab}_\Gamma(hP)$ has a countable image, whence must be essentially trivial (see e.g. [13, Proposition 1.3]), and thus the Γ -action on $(H/P, \nu_o)$ is essentially free.

It remains to prove that $(H/P, \nu_o)$ is a *prime* (Γ, τ) -space. By Corollary 4.15 this amounts to showing that P is a maximal proper closed subgroup of H . The *Bruhat decomposition* of H with respect to P tells us that

$$H = P \sqcup PwP, \quad \text{where} \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This decomposition demonstrates two things. Firstly, the group generated by P and w equals H . Secondly, any subgroup which strictly contains P must contain the element w , whence also the group generated by P and w . In particular, P is a maximal (not necessarily closed) subgroup of H , and we are done.

8. PROOF OF COROLLARY 1.12

Let (Γ, Λ) be a Hecke pair. We shall assume that Λ is non-amenable and not co-amenable in Γ . The latter assumption guarantees by [2, Proposition 3.4] that there exists a completion triple (H, L, ρ) of (Γ, Λ) with H non-amenable.

Let us fix a spread-out Λ -absorbing probability measure on Γ . Since Λ is non-amenable, so is Γ , and thus the Poisson boundary (B_τ, ν_τ) of (Γ, τ) is non-trivial. Let (H, θ_τ) denote the Hecke completion of (Γ, τ) with respect to ρ . Since H is non-amenable, the Poisson boundary $(B_{\theta_\tau}, \nu_{\theta_\tau})$ of (H, θ_τ) is non-trivial. By (I) in Corollary 1.9, $(B_{\theta_\tau}, \nu_{\theta_\tau})$ is thus a non-trivial τ -boundary. Since Λ is non-amenable, Corollary 1.9 (III) tells us (B_τ, ν_τ) and $(B_{\theta_\tau}, \nu_{\theta_\tau})$ cannot be measurably Γ -isomorphic. We conclude that (Γ, τ) is not a prime measured group.

9. PROOF OF THEOREM 1.13

Let $(K, |\cdot|)$ be a non-Archimedean local field with a prime residue field, and fix a uniformizer ϖ of K . We retain the notation from Subsection 1.5. In particular,

$$\Gamma = \Xi \rtimes \langle \varpi \rangle \quad \text{and} \quad \Lambda = \Xi_o \rtimes \{1\},$$

and

$$H = K \rtimes \langle \varpi \rangle \quad \text{and} \quad L = \mathcal{O} \rtimes \{1\},$$

where Ξ_o and Ξ are the additive subgroups of K generated by all powers of ϖ and all positive powers of ϖ respectively. We shall think of Γ as a dense subgroup of H , and we recall that H acts transitively on K via

$$(x, \varpi^n)y = x + \varpi^n y, \quad \text{for } (x, \varpi^n) \in H \text{ and } y \in K,$$

so that $K \cong H/P$, with $P = \{0\} \rtimes \langle \varpi \rangle$.

Lemma 9.1. *P is a maximal closed subgroup of H .*

Proof. Suppose that Q is a proper closed subgroup which strictly contains P . Then there exists a non-zero $x \in K$ such that $P(x, 1)P \subset Q$, and thus

$$x\Xi \rtimes \langle \varpi \rangle \subset Q,$$

since Ξ is equal the additive group generated by all powers of ϖ . By assumption, Ξ is dense in K , whence the left-hand side is dense in H . Since Q is closed, we see that $Q = H$. \square

Throughout the rest of this section, we fix a finitely supported, spread-out and Λ -absorbing probability measure τ on Γ which satisfies

$$\sum_{n \in \mathbb{Z}} n (\text{pr}_{\mathbb{Z}})_* \tau(n) < 0, \tag{9.1}$$

where $\text{pr}_{\mathbb{Z}}$ is defined in (1.1). We write (H, θ_τ) for the Hecke completion of the Hecke measured group (Γ, Λ) , and note that since $\text{pr}_{\mathbb{Z}}$ is a homomorphism,

$$\sum_{n \in \mathbb{Z}} n (\text{pr}_{\mathbb{Z}})_* \theta_\tau(n) < 0, \tag{9.2}$$

by (P1) in Theorem 1.6.

9.1. Proofs of (I) and (II) and (III)

Since both Γ and H are non-exceptional subgroups of the affine group of K (in the sense of [21]), the supports of τ and θ_τ generate (as semi-groups) Γ and H respectively and (9.1) and (9.2) hold, [21, Theorem 2] tells us that

- (F1) there is a unique τ -stationary probability measure ν on K .
- (F2) there is a unique θ_τ -stationary probability measure ν' on K .
- (F3) ν is τ -proximal and ν' is θ_τ -proximal.

By (P3) in Theorem 1.6, ν' is also τ -stationary, whence $\nu = \nu'$ by uniqueness. Furthermore, since θ_τ is spread-out, ν is H -quasi-invariant by (I) in Lemma 3.1. Since H acts transitively on K and the Haar measure m_K on K is H -quasi-invariant, we conclude (from the uniqueness of H -invariant measure classes) that ν is absolutely continuous with respect to m_K . We can thus write $d\nu = u dm_K$ for some non-negative $u \in L^1(K, m_K)$. Since θ_τ is bi- L -invariant, and both ν and m_K are L -invariant, u is L -invariant and thus (almost everywhere equal to) a continuous function on K , which we still denote by u . We claim that u is strictly positive. If it were not, then since it is L -invariant, the zero set of u must be a non-empty \mathcal{O} -invariant set, and thus $\nu(\mathcal{O} + t) = 0$ for *some* $t \in K$. However, since ν is H -quasi-invariant, this readily implies that $\nu(\mathcal{O} + t) = 0$ for *all* $t \in K$. However, this leads to a contradiction. Indeed, since \mathcal{O} is an open (additive) subgroup K , the quotient space K/\mathcal{O} must be countable. In particular, we can find a countable set $T \subset K$ such that $K = \cup_{t \in T} (\mathcal{O} + t)$. Hence,

$$1 = \nu(K) = \nu\left(\bigcup_{t \in T} (\mathcal{O} + t)\right) \leq \sum_{t \in T} \nu(\mathcal{O} + t) = 0,$$

and thus we conclude that u is strictly positive. Finally, by Lemma 9.1, P is a maximal closed subgroup of H , and since $K \cong H/P$, Corollary 4.15 tells us that the Borel (Γ, τ) -space (K, ν) is prime.

9.2. Proof of (IV)

We begin with a few general remarks. Let G be a lcsc group and suppose that (X, \mathcal{B}_X) is a measurable space endowed with a jointly measurable action of G . Let ξ be a G -quasi-invariant σ -finite measure on \mathcal{B}_X . Fix $p \geq 1$ be a real number. The *regular L^p -representation* $(L^p(X, \xi), \sigma_{p, \xi})$ is given by

$$\sigma_{p, \xi}(g)f = \left(\frac{dg\xi}{d\xi}\right)^{1/p} f \circ g^{-1}, \quad \text{for } g \in G \text{ and } f \in L^p(X, \xi).$$

We say that $(L^p(X, \xi), \sigma_{p, \xi})$ is *L^p -irreducible* if for every non-zero $f \in L^p(X, \xi)$, the linear span of the set $\sigma_{\xi, p}(G)f$ is norm-dense in $L^p(X, \xi)$. Equivalently, $(L^p(X, \xi), \sigma_{p, \xi})$ is *not L^p -irreducible* if there exist non-zero $f_1 \in L^p(X, \xi)$ and $f_2 \in L^{p'}(X, \xi)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, such that

$$\int_X \left(\frac{dg\xi}{d\xi}\right)^{1/p} f_1 \circ g^{-1} \overline{f_2} d\xi = 0, \quad \text{for all } g \in G. \quad (9.3)$$

We note that (9.3) only depends on the measure class of ξ : suppose that ξ' is another G -quasi-invariant σ -finite measure on X , which is equivalent to ξ , and write $d\xi' = u d\xi$, for some (ξ -almost everywhere) strictly positive function u . Then,

$$F_1 := u^{-1/p} f_1 \in L^p(X, \xi') \quad \text{and} \quad F_2 := u^{-1/p'} f_2 \in L^{p'}(X, \xi'),$$

and for every $g \in G$,

$$\frac{dg\xi'}{d\xi'} = \frac{u \circ g^{-1}}{u} \frac{dg\xi}{d\xi}, \quad \xi\text{-almost everywhere,}$$

which readily implies that, for every $g \in G$,

$$\int_X \left(\frac{dg\xi'}{d\xi'} \right)^{1/p} F_1 \circ g^{-1} \overline{F_2} d\xi'$$

equals

$$\int_X \left(\frac{u \circ g^{-1}}{u} \right)^{1/p} \left(\frac{dg\xi}{d\xi} \right)^{1/p} (u \circ g^{-1})^{-1/p} f_1 \circ g^{-1} u^{-1/p} \overline{f_2} u d\xi$$

which simplifies to

$$\int_X \left(\frac{dg\xi}{d\xi} \right)^{1/p} f_1 \circ g^{-1} \overline{f_2} d\xi,$$

since $\frac{1}{p} + \frac{1}{p'} = 1$. In particular,

$$\int_X \left(\frac{dg\xi'}{d\xi'} \right)^{1/p} F_1 \circ g^{-1} \overline{F_2} d\xi' = \int_X \left(\frac{dg\xi}{d\xi} \right)^{1/p} f_1 \circ g^{-1} \overline{f_2} d\xi, \quad \text{for all } g \in G.$$

m_K -DECOUPLED PAIRS

Let $\Delta : G \rightarrow \mathbb{R}_+^*$ be a continuous homomorphism. We say that a σ -finite and G -quasi-invariant measure ξ' on X is Δ -invariant if

$$g\xi' = \Delta(g)\xi', \quad \text{for all } g \in G,$$

or equivalently, if $\frac{dg\xi'}{d\xi'} = \Delta(g)$, almost everywhere. It follows from the discussion above that if ξ is equivalent to a Δ -invariant σ -finite measure ξ' , then $(L^p(X, \xi), \sigma_{p, \xi})$ is *not* L^p -irreducible if and only if there are non-zero $F_1 \in L^p(X, \xi')$ and $F_2 \in L^{p'}(X, \xi')$ such that

$$\int_X F_1 \circ g^{-1} \overline{F_2} d\xi' = 0, \quad \text{for all } g \in G.$$

Let us from now on assume that X is a locally compact and second countable space, and that G acts jointly continuously on X and there is a Δ -invariant σ -finite Borel measure ξ' on X , which is equivalent to ξ . We say that a pair (F_1, F_2) of non-zero *compactly supported* continuous functions on X are ξ' -decoupled if

$$\int_X F_1 \circ g^{-1} \overline{F_2} d\xi' = 0, \quad \text{for all } g \in G.$$

Since F_1 and F_2 are compactly supported and continuous, they belong to $L^p(K, \xi')$ for every $p \in [1, \infty]$. The following lemma is immediate from our discussion above.

Lemma 9.2. *If a ξ' -decoupled pair exists, then $(L^p(X, \xi), \sigma_{\xi, p})$ is not L^p -irreducible for any $p \in [1, \infty)$.*

Let us now specialize to the setting of Theorem 1.13, where

$$G = H \quad \text{and} \quad X = K \quad \text{and} \quad \xi = \nu \quad \text{and} \quad \xi' = m_K,$$

where ν is the unique τ -stationary and θ_τ -stationary probability measure on K (and equivalent to m_K by (II)).

Set $\Delta_K(x, \varpi^n) = q^n$, where $q = |\mathcal{O}/\mathcal{P}|$ is a prime number. A straightforward calculation shows that m_K is Δ_K -invariant. To prove Theorem 1.13, it suffices by Lemma 9.2 to show

that (K, m_K) admits a m_K -decoupled pair. This is done in the following lemma. Let K^* denote the multiplicative group of K , and write

$$\mathcal{O}^* = \{x \in \mathcal{O} \mid |x| = 1\}$$

for the multiplicative group of the open sub-ring $\mathcal{O} \subset K$, and \widehat{K} for the (additive) dual of K , i.e. the multiplicative group of continuous homomorphisms from $(K, +)$ into the circle group $\mathbb{S}^1 \subset \mathbb{C}^*$.

Lemma 9.3. *For all $z_1, z_2 \in \mathcal{O}^*$ and $\lambda \in \widehat{K}$ such that*

$$|z_1 - z_2| \geq \frac{1}{q} \quad \text{and} \quad \lambda|_{\varpi\mathcal{O}} \neq 1,$$

we have

$$\int_K F_{z_1}(y + \varpi^m x) \overline{F_{z_2}(x)} dm_K(x) = 0, \quad \text{for all } (y, \varpi^m) \in H,$$

where $F_z(x) = \lambda(xz) \chi_{\mathcal{O}}(x)$ for $z \in K^*$. In particular, (F_{z_1}, F_{z_2}) is a m_K -decoupled pair.

Remark 9.4. For example, the assumptions on z_1 and z_2 are satisfied for

$$z_1 = 1 \quad \text{and} \quad z_2 = 1 + \varpi.$$

Furthermore, if λ' is a non-trivial character on K (which always exists), then it must be non-trivial on some subgroup of K of the form $\varpi^{-N}\mathcal{O}$ for some $N \geq 0$ (since these subgroups exhaust K). If we set

$$\lambda(x) = \lambda'(x\varpi^{-(N+1)}), \quad \text{for } x \in K,$$

then λ is a character on K which is non-trivial on $\varpi\mathcal{O}$.

Proof. We first note that for all $z \in K$ and $\lambda \in \widehat{K}$,

$$F_z(y + \varpi^m x) = \lambda(yz) \lambda(x\varpi^m z) \chi_{\varpi^{-m}(\mathcal{O}-y)}(x), \quad \text{for all } y \in K \text{ and } m \in \mathbb{Z},$$

whence, for all $y, z_1, z_2 \in K$ and $m \in \mathbb{Z}$,

$$\begin{aligned} \int_K F_{z_1}(y + \varpi^m x) \overline{F_{z_2}(x)} dm_K(x) &= \lambda(yz_1) \int_K \lambda(x(\varpi^m z_1 - z_2)) \chi_{\varpi^{-m}(\mathcal{O}-y) \cap \mathcal{O}}(x) dm_K(x) \\ &= \lambda(yz_1) \int_{\mathcal{O}_{y,m}} \lambda(xw_m) dm_K(x), \end{aligned}$$

where

$$\mathcal{O}_{y,m} = \varpi^{-m}(\mathcal{O} - y) \cap \mathcal{O} \quad \text{and} \quad w_m = \varpi^m z_1 - z_2.$$

In what follows, we shall assume that

$$z_1, z_2 \in \mathcal{O}^* \quad \text{and} \quad |z_1 - z_2| \geq \frac{1}{q} \quad \text{and} \quad \lambda|_{\varpi\mathcal{O}} \neq 1,$$

and show that (F_{z_1}, F_{z_2}) is a m_K -decoupled pair. Note that the first two assumptions imply that $w_m \neq 0$ for all $m \in \mathbb{Z}$.

To prove the lemma, it clearly suffices to show that

$$\int_{\mathcal{O}_{y,m}} \lambda_m(x) dm_K(x) = 0, \quad \text{for all } y \in K \text{ and } m \in \mathbb{Z}, \quad (9.4)$$

where λ_m is the (non-trivial) character on K given by $\lambda_m(x) = \lambda(w_m x)$, for $x \in K$.

If $\mathcal{O}_{y,m} = \emptyset$, then (9.4) is automatic. Note that

$$\mathcal{O}_{y,m} \neq \emptyset \iff y \in \mathcal{O} - \varpi^m \mathcal{O} = \begin{cases} \mathcal{O} & \text{if } m \geq 0 \\ \varpi^m \mathcal{O} & \text{if } m < 0 \end{cases}.$$

We shall deal with the cases $m \geq 0$ and $m < 0$ separately.

CASE I: $m \geq 0$ AND $y \in \mathcal{O}$

In this case, $\mathcal{O}_{y,m} = \mathcal{O}$. To prove (9.4), it suffices to show that $\lambda_m|_{\mathcal{O}} \neq 1$ (since the Haar-integral of any non-trivial character on a compact abelian group is zero). Note that

$$\lambda_m|_{\mathcal{O}} \neq 1 \iff \lambda|_{w_m \mathcal{O}} \neq 1.$$

Since $\lambda|_{\varpi \mathcal{O}} \neq 1$, it thus suffices to show that $\varpi \mathcal{O} \subseteq w_m \mathcal{O}$, or equivalently, $|w_m| \geq q^{-1}$. The case $m = 0$ holds by assumption; indeed $|w_0| = |z_1 - z_2| \geq q^{-1}$. If $m \geq 1$, then since $z_1, z_2 \in \mathcal{O}^*$, and thus $|z_1| = |z_2| = 1$, we have

$$1 = |z_2| = |w_m - \varpi^m z_1| \leq \max(|w_m|, \underbrace{|\varpi^m z_1|}_{<1}) = |w_m|,$$

by the ultra-metric triangle inequality.

CASE II: $m < 0$ AND $y \in \varpi^m \mathcal{O}$

In this case,

$$\begin{aligned} \mathcal{O}_{y,m} &= \varpi^{-m}(\mathcal{O} - y) \cap \mathcal{O} = \varpi^{-m}((\mathcal{O} - y) \cap \varpi^m \mathcal{O}) \\ &= \varpi^{-m}(\mathcal{O} \cap (\varpi^m \mathcal{O} + y) - y) = \varpi^{-m}(\mathcal{O} \cap \varpi^m \mathcal{O} - y) \\ &= \varpi^{-m}(\mathcal{O} - y) = \varpi^{-m} \mathcal{O} - \varpi^{-m} y. \end{aligned}$$

Hence,

$$\int_{\mathcal{O}_{y,m}} \lambda_m(x) dm_K(x) = \lambda_m(-\varpi^{-m} y) \int_{\varpi^{-m} \mathcal{O}} \lambda_m(x) dm_K(x),$$

and thus to establish (9.4), we only need to show that $\lambda_m|_{\varpi^{-m} \mathcal{O}} \neq 1$ (since the Haar-integral of any non-trivial character on a compact abelian group is zero). Note that

$$\lambda_m|_{\varpi^{-m} \mathcal{O}} \neq 1 \iff \lambda|_{w_m \varpi^{-m} \mathcal{O}} \neq 1.$$

Since $\lambda|_{\varpi \mathcal{O}} \neq 1$, it thus suffices to establish the inclusion $\varpi \mathcal{O} \subseteq w_m \varpi^{-m} \mathcal{O}$, or equivalently, the lower bound $|w_m| \geq q^{-(m+1)}$. However, since $m < 0$ and $|z_1| = |z_2| = 1$, we have

$$1 < q^{-m} = |\varpi^m z_1| = |w_m + z_2| \leq \max(|w_m|, \underbrace{|z_2|}_{=1}) = |w_m|,$$

by the ultra-metric triangle inequality. □

Remark 9.5. The non- L^2 -irreducibility of $\sigma_{m_K,2}$ can be proved abstractly by appealing to the Mackey Little Group Method, developed in the seminal paper [49], according to which it suffices (in order to establish IV) to show that the multiplicative group $\langle \varpi \rangle$ does not act ergodically on \widehat{K} . Indeed, any $\langle \varpi \rangle$ -invariant measurable subset, which is neither null nor co-null with respect to $m_{\widehat{K}}$, provides a non-trivial H -equivariant projection on $L^2(K, m_K)$ (which in particular implies that $\sigma_{m_K,2}$ is not L^2 -irreducible). However, it is not immediately clear that this projection extends continuously to other L^p -spaces, which is why we have preferred to outline the more constructive approach above.

10. PROOF OF THEOREM 1.19

Throughout this section, we shall fix

- a discrete countable *prime* measured group (Ξ, σ_o) with finite entropy.
- a positive summable sequence $\beta = (\beta_k)$.

Our aim is to construct from σ_o a probability measure τ_β on the direct sum

$$\Gamma := \bigoplus_{\mathbb{N}} \Xi = \bigcup_{n=1}^{\infty} \left(\prod_{k=1}^n \Xi \right),$$

(with the obvious inclusions), whose support generates Γ as a semi-group, such that

$$\text{BndEnt}(\Gamma, \tau_\beta) = \left\{ \sum_{k \in S} \beta_k \mid S \subseteq \mathbb{N} \right\}.$$

The following lemma, which will be proved below, provides all of the necessary ingredients in the construction of τ_β .

Lemma 10.1. *There exist*

- (I) *a probability measure σ on Ξ with finite entropy $h(\sigma)$, whose support generates Ξ as a semi-group, such that $\sigma_o * \sigma = \sigma * \sigma_o$,*
- (II) *a summable sequence (p_k) such that $p_k \in (0, 1)$ for all k ,*
- (III) *a positive sequence (q_k) such that*

$$1 = q_1 > q_2 > \dots \quad \text{and} \quad \lim_k q_k = 0,$$

so that $\beta_k = p_k q_k h(\sigma)$ for all k .

10.1. Proof of Theorem 1.19 assuming Lemma 10.1

We denote by (B_o, ν_o) the Poisson boundary of (Ξ, σ_o) . Since $\sigma * \sigma_o = \sigma_o * \sigma$, (II) in Proposition 4.7 tells us that (B_o, ν_o) is also a maximal σ -proximal Borel (Ξ, σ) -space, which is prime by assumption.

For $k \geq 1$, we define $\sigma_k \in \text{Prob}(\Xi)$ by

$$\sigma_k = (1 - p_k)\delta_e + p_k\sigma.$$

Since $\sigma * \sigma_k = \sigma_k * \sigma$, (II) Proposition 4.7 again tells us that (B_o, ν_o) is a maximal σ_k -proximal Borel (Ξ, σ_k) -space, whence the Poisson boundary of (Ξ, σ_k) . Furthermore, by Lemma 4.16,

$$h_k := h_{\sigma_k}(B_o, \nu_o) = p_k h_\sigma(B_o, \nu_o) = p_k h(\sigma), \quad \text{for all } k.$$

We now define the probability measure $\tilde{\tau}$ on the direct product $\prod_{\mathbb{N}} \Xi$ by

$$\tilde{\tau} = \sigma_1 \otimes \sigma_2 \otimes \dots,$$

Since (p_k) is summable and $\sigma_k(e) \geq 1 - p_k$ for all k ,

$$\tilde{\tau}(e, e, \dots) = \prod_{k=1}^{\infty} \sigma_k(e) \geq \prod_{k=1}^{\infty} (1 - p_k) > 0.$$

By [7, Section 3.V], this implies that $\tilde{\tau}$ gives full measure to $\Gamma < \prod_{\mathbb{N}} \Xi$. Since each σ_k generates Ξ as a semigroup, we see that $\tilde{\tau}$ is a spread-out probability measure on Γ .

Since (B_o, ν_o) is a prime Borel (Ξ, σ_k) -space for every k , the following proposition is now a special case of [7, Theorem 3.6].

Proposition 10.2. *Every $\tilde{\tau}$ -boundary is of the form*

$$(B_I, \nu_I) := \prod_{k \in I} (B_o, \nu_o), \quad \text{for some } I \subseteq \mathbb{N}.$$

In particular, the Poisson boundary of $(\Gamma, \tilde{\tau})$ is $(B_{\mathbb{N}}, \nu_{\mathbb{N}})$.

Let us now turn to the construction of τ_β . We recall from Lemma 10.1 that

$$\beta_k = p_k q_k h(\sigma), \quad \text{for all } k \geq 1. \quad (10.1)$$

For $n \geq 1$, we set $\alpha_n = q_n - q_{n+1} > 0$, so that

$$\sum_{n=1}^{\infty} \alpha_n = 1 \quad \text{and} \quad \sum_{n=k}^{\infty} \alpha_n = q_k \quad \text{for all } k \geq 1, \quad (10.2)$$

and define $\tau_\beta \in \text{Prob}(\Gamma)$ by

$$\tau_\beta = \sum_{n=1}^{\infty} \alpha_n \sigma_1 \otimes \cdots \otimes \sigma_n,$$

with the obvious interpretation of each term as a probability measure on Γ . Clearly,

$$\tau_\beta * \tilde{\tau} = \tilde{\tau} * \tau_\beta,$$

so by (II) in Proposition 4.7, every $\tilde{\tau}$ -boundary is a τ_β -boundary (and vice versa). Hence, by Proposition 10.2

$$\text{BndEnt}(\Gamma, \tau_\beta) = \left\{ h_{\tau_\beta}(B_I, \nu_I) \mid I \subseteq \mathbb{N} \right\}.$$

It remains for us to compute $h_{\tau_\beta}(B_I, \nu_I)$ for every $I \subseteq \mathbb{N}$. By Lemma 4.16 and after some simple rearrangements and applying (10.1) and (10.2), we see that

$$\begin{aligned} h_{\tau_\beta}(B_I, \nu_I) &= \sum_{n=1}^{\infty} \alpha_n h_{\sigma_1 \otimes \cdots \otimes \sigma_n}(B_{I \cap [1, n]}, \nu_{I \cap [1, n]}) = \sum_{n=1}^{\infty} \alpha_n \left(\sum_{k \in I \cap [1, n]} h_k \right) \\ &= \sum_{k \in I} \left(\sum_{n=k}^{\infty} \alpha_n \right) h_k = \sum_{k \in I} p_k q_k h(\sigma) = \sum_{k \in I} \beta_k, \end{aligned}$$

which finishes the proof.

10.2. Proof of Lemma 10.1

We begin by stating the following simple lemma, whose proof is left to the reader.

Lemma 10.3. *For every positive summable sequence (β_k) , there is a strictly increasing positive sequence (w_k) with $w_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $\sum_{k=1}^{\infty} \beta_k w_k < \infty$.*

Let us now fix a positive summable sequence (β_k) , as well as a strictly increasing positive sequence (w_k) as in the lemma above. Upon scaling the sequence (w_k) we can clearly ensure that

$$\sum_{k=1}^{\infty} \beta_k w_k < \infty \quad \text{and} \quad \beta_k w_k \in (0, 1) \quad \text{for all } k \geq 1.$$

Given a probability measure σ on Ξ with finite positive entropy $h(\sigma)$, we now set

$$p_k = \beta_k w_k \quad \text{and} \quad q_k = \frac{1}{h(\sigma) w_k}.$$

Then (q_k) is a strictly decreasing positive sequence with $q_k \rightarrow 0$ as $k \rightarrow \infty$, and thus to establish Lemma 10.1 it remains to prove that we can always find $\sigma \in \text{Prob}(\Xi)$ such that

$$\sigma * \sigma_o = \sigma_o * \sigma \quad \text{and} \quad h(\sigma) = \frac{1}{w_1}.$$

To do this, let us define

$$\sigma_{\varepsilon, N} = (1 - \varepsilon)\delta_e + \varepsilon\sigma_o^{*N}, \quad \text{for } \varepsilon \in (0, 1) \text{ and } N \geq 1.$$

By Lemma 4.16, $h(\sigma_{\varepsilon, N}) = \varepsilon N h(\sigma_o)$. If we choose ε and N so that

$$\varepsilon N h(\sigma_o) = \frac{1}{w_1},$$

we see that $h(\sigma_{\varepsilon, N}) = \frac{1}{w_1}$. Clearly $\sigma := \sigma_{N, \varepsilon}$ is spread-out (since σ_o is spread-out), and

$$\sigma * \sigma_o = \sigma_o * \sigma,$$

which finishes the proof.

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