

# SOME COMMENTS TO MAGNETIC FIELD REPRESENTATION FOR BEAM DYNAMIC CALCULATIONS

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## Abstract

Machines with high currents and small apertures, as used for SIS100 of the FAIR project, require a sincere understanding of the resonances excited by the magnetic field distortions; typically performed by tracking codes. These codes model the field errors using a Taylor Series approximation of the field quality at the track of the ideal particle. The path of the particle within the elliptic aperture of the dipole is curved; thus the standard approach of using plane circular multipoles fails to model the real symmetry of the magnetic field, an important feature of effective field description for beam loss calculations. Therefore toroidal elliptic multipoles were developed which allow describing the magnetic field concisely in an elliptic vacuum chamber in curved dipoles and quadrupoles.

## INTRODUCTION

SIS100, the core component of the FAIR accelerator, uses superconducting fast ramped magnets. These dipole magnets are curved and the beam aperture is elliptic.

The precise field of an accelerator magnet can only be found by numerical calculations. Analytic expressions are fitted to this field in the aperture in order to distribute information on this field in a convenient and concise way. Particular solutions of the potential equation are used as basis functions for this approach. The standard tool are plane circular multipoles. Plane elliptic multipoles are useful if the beam aperture has an elliptic cross section [1, 2, 3]. Their advantages are a larger reference area, an ellipse surrounding the reference circle of the former case, and better convergence properties. In a curved magnet curvature effect are not entirely negligible. Toroidal circular multipoles were developed [4] and demonstrated that the coefficients can be obtained by a rotating coil probe [5, 6]. Now elliptic toroidal multipoles have been developed. In this paper we present how these are to be calculated.

## TOROIDAL ELLIPTIC MULTIPOLES

In the aperture of a curved magnet the reference volume is a segment of a torus. Its vertical cross section is a circle or an ellipse. In the latter case toroidal elliptic coordinates are the appropriate tool.

### Toroidal Elliptic Coordinates

These orthogonal coordinates are obtained by shifting a plane vertical ellipse by an amount  $R_c$  along the X-axis off

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the origin and then rotating it by the azimuth  $\phi$  around the original vertical (Z-) axis:

$$X + iY = (R_c + e \cosh \eta \cos \psi) e^{i\phi} \quad (1)$$

$$Z = e \sinh \eta \sin \psi. \quad (2)$$

$R_c$  is curvature radius. The semi-axes  $a, b$  determine the eccentricity  $e$  and the value  $\eta = \eta_0$  characterising the reference ellipse.  $e/R_c$  is the inverse aspect ratio  $\bar{\epsilon}$ :

$$e = \sqrt{a^2 - b^2} \quad \tanh(b/a) = \eta_0. \quad (3)$$

$$\bar{\epsilon} = e/R_c \quad (4)$$

The metric coefficients are:

$$h_t(\eta, \psi) = h_\eta = h_\psi = e \sqrt{\cosh^2 \eta - \cos^2 \psi} \quad (5)$$

$$h_\phi(\eta, \psi) = R_c h(\eta, \psi) = R_c (1 + \bar{\epsilon} \cosh \eta \cos \psi) \quad (6)$$

$$h_{t0} = h_t(\eta_0, \psi), \quad h_0 = h(\eta_0, \psi). \quad (7)$$

### The Potential

Only fields and potentials uniform in  $\phi$  are considered. So these quantities are the same in each cross section  $\phi = \text{const.}$ ; they are independent of  $\phi$ . The potential equation is:

$$\frac{1}{h_t^2} \left[ \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \psi^2} + \right. \quad (8)$$

$$\left. + \frac{\bar{\epsilon}}{h} \left( \sinh \eta \cos \psi \frac{\partial}{\partial \eta} - \cosh \eta \sin \psi \frac{\partial}{\partial \psi} \right) \right] \Phi = 0$$

$$\frac{1}{h_t^2 \sqrt{h}} \left[ \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \psi^2} + \frac{\bar{\epsilon}^2}{4h^2} \right] (\sqrt{h} \Phi) = 0. \quad (9)$$

$\bar{\epsilon}$  is small; so neglecting second and higher powers gives a good approximation. The operator then remaining in (8) is just the Laplacian in plane elliptical coordinates  $\eta, \psi$ . A complete set of particular solutions of the latter equation is:  $\{1, \cosh(n\eta) \cos(n\psi), \sinh(n\eta) \sin(n\psi)\}$ ,  $n = 1, 2, 3 \dots$ . Thus an approximate solution of (9) may be written as:

$$\Phi(\eta, \psi) = B_0 h^{-1/2} \left[ a_0 + \right. \quad (10)$$

$$\left. \sum_{n=1}^M \left[ \frac{a_n \cosh(n\eta)}{n \cosh(n\eta_0)} \cos(n\psi) + \frac{b_n \sinh(n\eta)}{n \sinh(n\eta_0)} \sin(n\psi) \right] \right]$$

### Determining The Coefficients

The coefficients  $a_0, a_1, \dots, b_1, b_2$  can be determined from the Cartesian field components  $B_y(\psi), B_x(\psi)$  given on the

reference ellipse. They are either found numerically from a field code or experimentally by measurements. Now we use a local two-dimensional coordinate system in the cross section  $\phi = 0$ . Its origin is at  $(X, Y, Z) = (R_c, 0, 0)$ ; the x-axes are parallel; the y-axis is parallel to the Z-axis.

$$\mathbf{r}_\ell = e (\cosh \eta \cos \psi, \sinh \eta \sin \psi) \quad (11)$$

The reference ellipse  $\mathcal{E}$  and all values characterising it are the same as above. The unit vector normal to  $\mathcal{E}$ , i.e.  $\mathbf{n}_\eta$  is:

$$\mathbf{n}_\eta(\eta, \psi) = \frac{d\mathbf{r}_\ell}{d\eta} / \left| \frac{d\mathbf{r}_\ell}{d\eta} \right|, \quad \mathbf{n}_{\eta_0}(\psi) = \mathbf{n}_\eta(\eta_0, \psi). \quad (12)$$

The component  $B_\eta$  on  $\mathcal{E}$  is:

$$B_{\eta_0}(\psi) = B_\eta|_{\eta=\eta_0} = \mathbf{n}_{\eta_0}(\psi) \cdot \vec{B}|_{\eta=\eta_0} = \mathbf{n}_{\eta_0 x} B_x + \mathbf{n}_{\eta_0 y} B_y \quad (13)$$

or

$$\begin{aligned} \chi(\psi) := & - \frac{h_{t0} h_0^{3/2} B_{\eta_0}(\psi)}{B_0} = \quad (14) \\ & - e \sqrt{\cosh^2 \eta_0 - \cos^2 \psi} [1 + \bar{\epsilon} \cosh \eta_0 \cos \psi]^{3/2} \times \\ & \frac{B_x(\psi) \sinh \eta_0 \cos \psi + B_y(\psi) \cosh \eta_0 \sin \psi}{B_0 \sqrt{\sinh^2 \eta_0 \cos^2 \psi + \cosh^2 \eta_0 \sin^2 \psi}} \end{aligned}$$

On the other hand, from the series (10) and with

$$\vec{B} = - \text{grad} \Phi = - \frac{1}{h_t} \left( \frac{\partial \Phi}{\partial \eta}, \frac{\partial \Phi}{\partial \psi} \right) \quad (15)$$

we get:

$$\begin{aligned} -\chi(\psi) = & h_0^{3/2} \frac{\partial}{\partial \eta} (\Phi(\eta, \psi) / B_0) |_{\eta=\eta_0} = \\ = & h_0 \sum_{n=1}^M [a_n \tanh(n\eta_0) \cos(n\psi) + \\ & + b_n \coth(n\eta_0) \sin(n\psi)] \\ & - \frac{\bar{\epsilon}}{2} a_0 \sinh \eta_0 \cos \psi \\ & - \frac{\bar{\epsilon}}{2} \sinh \eta_0 \sum_{n=1}^M \left[ \frac{a_n}{n} \cosh(n\eta_0) \cos(n\psi) + \right. \\ & \left. + \frac{b_n}{n} \sinh(n\eta_0) \sin(n\psi) \right] \cos \psi \quad (16) \end{aligned}$$

## CONSTRUCTING THE CONVERSION MATRICES

Based on the function  $\chi(\psi)$  coefficients are to be calculated using

$$C_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \chi(\psi) \cos(n\psi) d\psi \quad (17)$$

and

$$D_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \chi(\psi) \sin(n\psi) d\psi \quad (18)$$

These have then to be converted to the appropriate local elliptic toroidal multipoles using

$$FA_n = C_n \quad (19)$$

and

$$GB_n = D_n \quad (20)$$

The conversion matrices  $F$  and  $G$  have been calculated using Mathematica<sup>TM</sup>. Up to now the calculations were performed up to coefficients of order 5. The given results show that these matrices can be constructed by simple formulae.

The matrix  $G$  consists of

$$G = CTH^0 + \epsilon (SH + CTH_1). \quad (21)$$

The matrix  $CTH^0$  is given by

$$CTH^0 = I_{m,m} 1 / \tanh((m-1)\eta_0), \quad (22)$$

with  $I$ , the identity matrix. The matrix  $SH$  consists of two bands. The lower band  $T^1$  is given by

$$T_{nm}^1 = \begin{cases} 1/(4(m-1)) & m = n+1 \\ 0 & m \neq n+1 \end{cases} \quad (23)$$

The upper band  $T^2$  is then given by

$$T_{nm}^2 = \begin{cases} 1/(4n) & m = n-1 \\ 0 & m \neq n-1 \end{cases} \quad (24)$$

So the matrix  $SH$  is then given by

$$SH = -\sinh(\eta_0) (T_1 + T_2) \quad (25)$$

The matrix  $CTH_1$  is again a banded matrix with its lower band  $CTH^1$  given by

$$CTH_{nm}^1 = \begin{cases} \coth((m-1)\eta_0) & m = n+1 \\ 0 & m \neq n+1 \end{cases} \quad (26)$$

and its upper band  $CTH^2$  given by

$$CTH_{nm}^2 = \begin{cases} \coth(n\eta_0) & m = n-1 \\ 0 & m \neq n-1 \end{cases} \quad (27)$$

thus  $CTH_1 = CTH^1 + CTH^2$ .

The matrix  $F$  one has to take the element  $a_0$  into account, thus its has one more row and column as matrix  $G$ . The matrix  $F$  is given by

$$F = TH_0 + \bar{\epsilon} (SH^0 + TH_1). \quad (28)$$

The matrix  $TH_0$  is given by

$$TH_0 = +I_{nn} \tanh((n-1)\eta_0). \quad (29)$$

The matrix  $SH^0$  is identical to  $SH$  if using the rule

$$SH_{n+1,m+1}^0 = SH_{n,m}. \quad (30)$$

All elements of  $SH_{m,1}^0$  are 0 except  $SH_{2,1}^0 = -1/2\bar{\epsilon}\sinh(\eta_0)$  and all elements of  $SH_{1,m}^0$  except that the element of  $SH_{1,2}^0 = 1/2\bar{\epsilon}\sinh(\eta_0)$ . The matrix  $TH_1$  is given by

$$TH_1 = \frac{1}{2}\bar{\epsilon} \cosh \eta_0 (TH^1 + TH^2), \quad (31)$$

which consists of two bands.  $TH^1$  is given by

$$TH_{nm}^1 = \begin{cases} \coth(m\eta_0) & m = n - 1 \text{ and } m > 1 \\ 0 & \text{others} \end{cases}. \quad (32)$$

The matrix  $TH^2$  is given by

$$TH_{nm}^2 = \begin{cases} \coth((m-2)\eta_0) & m = n + 1 \text{ and } m > 1 \\ 0 & \text{others} \end{cases}. \quad (33)$$

Constructing these matrices the vectors  $A_n = a_0, a_1, a_2, \dots$  and  $B_n = b_1, b_2, \dots$  can be calculated using numerical methods.

## CONCLUSION AND OUTLOOK

The results presented here show that now the magnetic field within an elliptic toroidal can be described concisely. The formulae given here have to be extended so that the field components  $B_y$  and  $B_x$  can be interpolated directly. This will then allow representing the field following the curvature of the beam. In next steps we will then use the measured data to obtain the coefficients  $a_n$  and  $b_n$  to study the difference in field reconstructed by the local elliptic toroidal multipoles and the plane circular multipoles typically used. The banded matrices show that a feed up and feed down of one coefficient to the next is to be expected. For typical accelerators the effect will be small as in curved magnets the dipole is the dominant term, thus mainly some spurious quadrupole is generated. This can be typically neglected as it will be much smaller than the quadrupole strength of the main quadrupole. The particular effect depends on the size of the beam and the machine radius.

The development given will allow interpreting the results of magnetic measurement and give a concise description of the magnetic field along a curved trajectory with elliptic dimensions.

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