



Institute of  
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# An alternative approach in deriving poroelastic plate theories

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**1** Motivation

**2** Poroelastic plates

**3** Numerical results

**4** Conclusion and Outlook

- Mathematical modelling of sound insulation by porous plates
- Mathematical modelling of the dynamical behaviour of poroelastic plates
- In classical theories, kinematic assumptions are introduced
  - → Kirchhoff plate
  - → Mindlin plate

- Mathematical modelling of sound insulation by porous plates
- Mathematical modelling of the dynamical behaviour of poroelastic plates
- In classical theories, kinematic assumptions are introduced
  - → Kirchhoff plate
  - → Mindlin plate
- Can the classical assumptions be transferred to poroelasticity, especially to the pore pressure?
- An assumption-free derivation is used

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- solid skeleton (including sealed pores)  
system of interconnected fluid infiltrated pores
- porosity :  $\phi = \frac{V^f}{V}$  ; full saturation assumed
- $\mathbf{u}$ ,  $p$  as degrees of freedom in the frequency domain

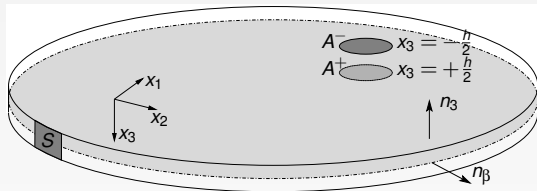
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system of interconnected fluid infiltrated pores
- porosity :  $\phi = \frac{V^f}{V}$  ; full saturation assumed
- $\mathbf{u}$ ,  $p$  as degrees of freedom in the frequency domain
- Total energy stored in the system :  $\Pi = \Pi(\mathbf{u}, p)$
- A variation from a state of equilibrium involves no change in energy

$$\delta\Pi = \int_{\Omega} \delta U_{\Omega} \, d\Omega + \int_{\Gamma} \delta U_{\Gamma} \, d\Gamma \stackrel{!}{=} 0$$

$$\delta\Pi = \delta\Pi(\mathbf{u}, \delta\mathbf{u}, p, \delta p) \quad \delta U = \delta U(\mathbf{u}, \delta\mathbf{u}, p, \delta p)$$

- An integration over the thickness coordinate is needed to deduce the plate equations

## ■ Geometry of the plate

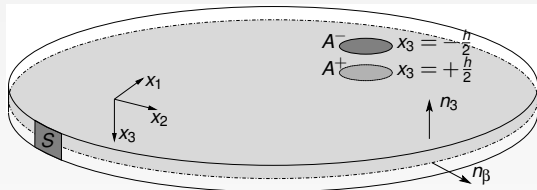


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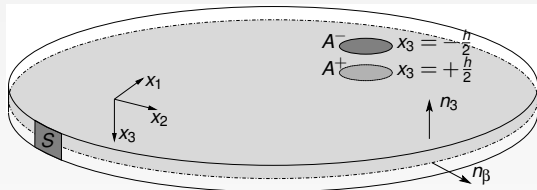
## ■ Substitution of $\mathbf{u}$ and $\rho$ by a power series in $x_3$ -direction

$$\mathbf{u}(x_1, x_2, x_3) = \sum_{k=0}^{\infty} \mathbf{u}^k(x_1, x_2) x_3^k$$

$$\rho(x_1, x_2, x_3) = \sum_{k=0}^{\infty} \rho^k(x_1, x_2) x_3^k$$

$\mathbf{u}^k(x_1, x_2), \rho^k(x_1, x_2) \dots$  Unknown functions of order  $k$

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$\delta\mathbf{u}^{\ell}(x_1, x_2), \delta\rho^{\ell}(x_1, x_2) \dots$  Unknown functions of order  $\ell$

- Integration over the plate thickness

$$\int_{\Omega} \delta U_{\Omega}(\mathbf{u}, \delta \mathbf{u}, \rho, \delta \rho) \, d\Omega$$

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- Extract plate problem (identify and decouple plate and shell quantities)

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- Extract plate problem (identify and decouple plate and shell quantities)
- Poroelastic case does not impinge on the decoupling of plate and disc problem
- Truncation of the power series

## ■ The plate quantities

- $u_3^k \rightarrow k := 0, 2, 4, \dots$  Vertical displacement of the cross section ( $w^k$ )
- $u_\alpha^k \rightarrow k := 1, 3, 5, \dots$  Rotation of the cross section ( $\psi_\alpha^k$ )
- $p^k \rightarrow k := 1, 3, 5, \dots$  Pore pressure distribution over the cross section

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- Truncation with respect to a specific order of  $k$

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## ■ Truncation with respect to a specific order of $k$

## ■ Truncation with respect to the order of the plate thickness $h$ plate parameter

$$(c^2)^n = \left(\frac{h^2}{12}\right)^n \quad n \in \mathbb{N}$$

- $n = 0 \rightarrow$  Theory of zeroth order
- $n = 1 \rightarrow$  Theory of first order
- $n = 2 \rightarrow$  Theory of second order

Zeroth order  $\mathcal{L}_{3 \times 3}^0 \mathbf{u} = \mathbf{f}$  with  $\mathbf{u} = [w, \Psi, \rho]^\top$

First order  $\mathcal{L}_{6 \times 6}^1 \mathbf{u} = \mathbf{f}$  with  $\mathbf{u} = [w, \Psi, \rho, w, \Psi, \rho]^\top$

Second order  $\mathcal{L}_{9 \times 9}^2 \mathbf{u} = \mathbf{f}$  with  $\mathbf{u} = [w, \Psi, \rho, w, \Psi, \rho, w, \Psi, \rho]^\top$

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→ rigid body motions

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→ Fourth order PDE

Second order  $\mathcal{L}_{9 \times 9}^2 \mathbf{u} = \mathbf{f}$  with  $\mathbf{u} = [w^0, \Psi^1, \rho^1, w^2, \Psi^3, \rho^3, w^4, \Psi^5, \rho^5]^\top$

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- The factor  $c^2$  has to be involved when reducing the system, e.g.

$$c^2(\nabla w^0 + \Psi^1) = O(c^4) \approx 0$$

$$\nabla w^0 + \Psi^1 = O(c^2)$$

Zeroth order  $\mathcal{L}_{3 \times 3}^0 \mathbf{u} = \mathbf{f}$  with  $\mathbf{u} = [\overset{0}{w}, \overset{1}{\Psi}, \overset{1}{\rho}]^\top$

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- The factor  $c^2$  has to be involved when reducing the system, e.g.

$$c^2(\nabla \overset{0}{w} + \overset{1}{\Psi}) = O(c^4) \approx 0$$

$$\nabla \overset{0}{w} + \overset{1}{\Psi} = O(c^2)$$

- Problems arise when trying to solve the full system right away  
→ Reducing before solving



Linear ansatz in  $k$        $\mathcal{L}_{3 \times 3} \mathbf{u} = \mathbf{f}$     with     $\mathbf{u} = [w^0, \psi^1, \rho^1]^\top$

Quadratic ansatz in  $k$      $\mathcal{L}_{4 \times 4} \mathbf{u} = \mathbf{f}$     with     $\mathbf{u} = [w^0, \psi^1, \rho^1, w^2]^\top$

Cubic ansatz in  $k$        $\mathcal{L}_{6 \times 6} \mathbf{u} = \mathbf{f}$     with     $\mathbf{u} = [w^0, \psi^1, \rho^1, w^2, \psi^3, \rho^3]^\top$

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→ eighth order PDE

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→ twelfth order PDE

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- System can be solved as a whole, without being reduced

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→ twelfth order PDE

- System can be solved as a whole, without being reduced
- At least a quadratic ansatz in  $k$  is needed to model a Kirchhoff-type equation containing the term  $\frac{E h^3}{12(1-\nu^2)} \Delta \Delta w$

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- The reduced *First Order Problem* with  $\overset{0}{w} := w$  and  $\overset{1}{p} := p$

$$\begin{bmatrix} D\Delta\Delta - h\omega^2\rho\beta & h(B_1\Delta - \beta) \\ i\omega h(B_1\Delta - \beta) & -hB_2 \end{bmatrix} \begin{bmatrix} w \\ p \end{bmatrix} = \begin{bmatrix} F \\ Q \end{bmatrix}$$

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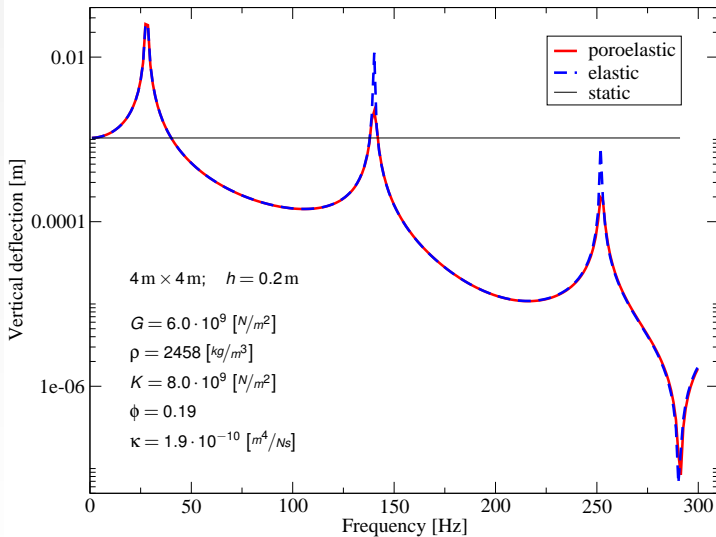
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- Weak form :

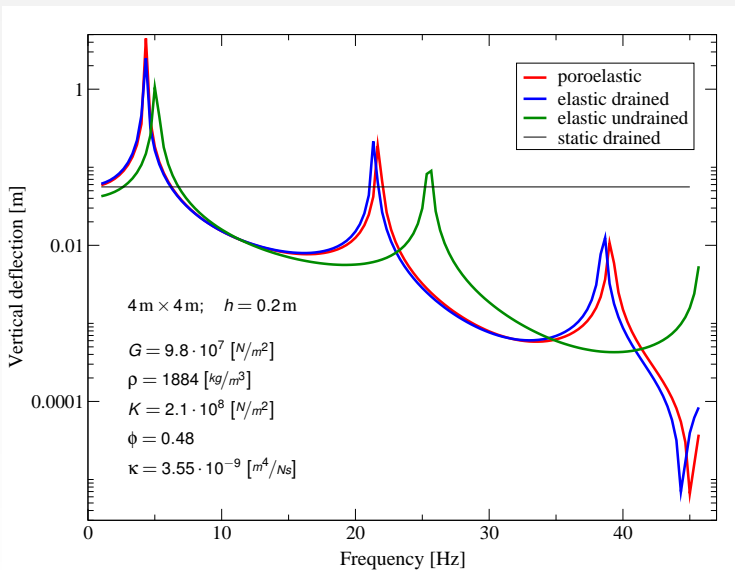
$$\int_A \left[ D[(1-\nu)\nabla\nabla w : \nabla\nabla v + \nu\Delta w\Delta v] - h\omega^2\rho\beta wv + h(B_1 p\Delta v - \beta pv) - Fv \right] dA -$$

$$\int_{\Gamma} \left[ V_n v - M_{nn} \frac{\partial v}{\partial n} \right] d\Gamma + [M_{ns} v]_{\bar{\mathbf{x}}}^{\bar{\mathbf{y}}} = 0 \quad \bar{\mathbf{x}}, \bar{\mathbf{y}} \in \Gamma$$

$$\int_A \left[ i\omega h(B_1\Delta wq - \beta wq) - hB_2 pq - Qq \right] dA = 0$$







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## ■ Conclusion

- Derivation of poroelastic plate equations by using series expansions in thickness direction
- Different treatment of the system depending on chosen kind of truncation
- Numerical solution of the *First Order Problem*

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## ■ Outlook

- Necessity to investigate higher order theories
- Analyse the full system concerning a stable numerical solution
- Compare the results to a 3D solution
- Coupling the plate with an acoustic fluid



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