SAMPLE VARIANCE IN FREE PROBABILITY

WIKTOR EJSMONT AND FRANZ LEHNER

Abstract. Let $X_1, X_2, \dots, X_n$ denote i.i.d. centered standard normal random variables, then the law of the sample variance $Q_n = \sum_{i=1}^n (X_i - \overline{X})^2$ is the χ^2-distribution with $n-1$ degrees of freedom. It is an open problem in classical probability to characterize all distributions with this property and in particular, whether it characterizes the normal law. In this paper we present a solution of the free analogue of this question and show that the only distributions, whose free sample variance is distributed according to a free χ^2-distribution, are the semicircle law and more generally so-called *odd* laws, by which we mean laws with vanishing higher order even cumulants. In the way of proof we derive an explicit formula for the free cumulants of Q_n which shows that indeed the odd cumulants do not contribute and which exhibits an interesting connection to the concept of R-cyclicity.

Dedicated to our friend and mentor Marek Bożejko on the occasion of his 70-th birthday

1. Introduction

Many questions in classical statistics involve characterization problems, which usually are instances of the following very general question:

Problem 1.1. Let $X_1, X_2, \dots, X_n$ be independent random variables with common unknown distribution function F, and $T := T(X_1, X_2, \dots, X_n)$ a statistic, based on $X_1, X_2, \dots, X_n$, with distribution function G. Can F be recovered from G?

Problems of this kind are the central leitmotiv of the fundamental work of Kagan, Linnik and Rao [17]. In the present paper we solve the free version of the following problem, which is still open in classical probability and might be called χ^2-conjecture, see [17, p. 466]:

Conjecture 1.2. *If $X_1, X_2, \dots, X_n$ are non-degenerate, independently and identically distributed classical random variables with finite non-zero variance σ^2, then a necessary and sufficient condition for X_1 to be normal is that $\sum_{i=1}^n (X_i - \overline{X})^2/\sigma^2$ be distributed as classical chi-square distribution with $n-1$ degrees of freedom.*

The classical χ^2-conjecture was studied previously by several authors. The first result is due to Ruben [28], who proved the conjecture under the assumption that either $n = 2$ or X_1 is symmetric. It is not known whether the symmetry hypothesis can be dropped for $n \geq 3$. In a later paper [29] Ruben used combinatorial tools to show that the symmetry condition can be dropped provided the sum of squares of the sample observations about the sample mean, divided by σ^2, is distributed as chi-square for two distinct sample sizes $m \neq n$ and $m, n \geq 2$. The proof given by Ruben is based on the cumulants of the sample variance and is somewhat complicated. Shortly later a simpler and more direct proof based on the moments of the sample variance was presented by Bondesson [5].

The original problem was solved recently by Golikova and Kruglov [13] under the additional assumption that $X_1, X_2, \dots, X_n$ are independent infinitely divisible random variables.

The following related characterization problem was solved by Kagan and Letac [18]: Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables and assume that

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the distribution of the quadratic statistic $\sum_{i=1}^{n}(X_i - \overline{X} + a_i)^2$ depends only on $\sum_{i=1}^{n} a_i^2$. Then each X_i have distribution $N(0,\sigma)$.

In the present paper we answer analogous questions in free probability. Free probability and free convolution was introduced by Voiculescu in [37] as a tool to study the von Neumann algebras of free groups. Free probability is now an established field of research with deep connections to combinatorics, random matrix theory, representation theory and many analogies to classical probability. Let us restrict our discussion to two specific ones, which are relevant to the problems discussed in the present paper. On the analytic side the Bercovici-Pata bijection [4] provides a one-to-one correspondence between infinitely divisible measures with respect to classical and free convolution. For example, the analogue of the normal law is played by Wigner's semicircle distribution which features as the limit law in the free central limit theorem.

On the combinatorial side we will make heavy use of free cumulants introduced by Speicher [31]. Roughly speaking, any result about classical cumulants can be translated to free probability by replacing the lattice of set partitions by the lattice of noncrossing partitions. Our standard reference for free cumulants is the book [27].

We are concerned here with free analogues of characterization theorems in the spirit of [17]. The study of free analogues of classical theorems has witnessed increasing interest during the last decade, see, e.g., [6, 12, 15, 22, 35, 33, 34]. Many properties of free random variables are analogous to those of their classical counterparts, in particular when they are picked according to the Bercovici-Pata bijection. There are, however, exceptions, mostly due to the failure of Marcinkiewicz' and Cramér's theorems in free probability. In particular, Bercovici and Voiculescu [3] showed that there exist free random variables with a finite number of nonvanishing free cumulants which are not semicircular, see [7] for a characterization of such distributions. This class of distributions appears in some (but not all) free characterization problems which are analogues of classical characterizations of the normal law, cf. [21, 7].

In the present paper we show that Conjecture 1.2 also falls in this class of problems and instead of Wigner laws we obtain the class of *odd* laws, i.e., laws with vanishing even cumulants. Such laws do not exist in classical probability, but can be constructed in free probability using the results of [7]. On the way we encounter a remarkable cancellation phenomenon: odd cumulants do not contribute to the distributions of certain quadratic statistics.

The paper is organized as follows. In section 2 we review basic free probability and the statement of the main result. Next in the subsection 2.2 we quote complementary facts, lemmas and indications. In the third section we prove our main results. Finally, in section 4 we look more closely at the relation between the sample variance, the free commutator, $R-$cyclic matrices and free infinite divisibility.

2. Free probability and statement of the main result

2.1. **Basic Notation and Terminology.** A tracial noncommutative probability space is a pair $(\mathcal{A},\tau)$ where $\mathcal{A}$ is a von Neumann algebra, and $\tau:\mathcal{A}\to \mathrm{C}$ is a normal, faithful, tracial state, i.e., τ is linear and continuous in the weak* topology, $\tau(XY)=\tau(YX)$, $\tau(\mathrm{I})=1$, $\tau(XX^*)\geq 0$ and $\tau(XX^*)=0$ implies $X=0$ for all $X,Y\in\mathcal{A}$.

The (usually taken to be self-adjoint) elements $X\in\mathcal{A}_{sa}$ are called (noncommutative) random variables. Given a noncommutative random variable $X\in\mathcal{A}_{sa}$, the distribution of X in the state τ is the unique probability measure μ_X on $\mathbb{R}$ (given by the spectral theorem) such that $\tau(f(X))=\int_{\mathbb{R}} f(\lambda)\,d\mu_X(\lambda)$ for any bounded Borel function f on $\mathbb{R}$.

2.2. **Free Independence, Free Convolution and Free infinite Divisibility.** A family of von Neumann subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$ are called *free* if $\tau(X_1\dots X_n)=0$ whenever $\tau(X_j)=0$ for all $j=1,\dots,n$ and $X_j\in\mathcal{A}_{i(j)}$ for some indices $i(1)\neq i(2)\neq\cdots\neq i(n)$. Random variables $X_1,\dots,X_n$ are freely independent (free) if the subalgebras they generate are free. Free random variables can be constructed using the reduced free product of von Neumann algebras [37]. For more details about free convolutions and free probability theory, the reader can consult [27, 36].

It can be shown that the joint distribution of free random variables X_i is uniquely determined by the distributions of the individual random variables X_i and therefore the operation of *free convolution* is well defined: Let μ and ν be probability measures on $\mathbb{R}$, and X, Y self-adjoint free random variables with respective distributions μ and ν, The distribution of $X+Y$ is called the free additive convolution of μ and ν and is denoted by $\mu \boxplus \nu$.
In analogy with classical probability, a probability measure μ on $\mathbb{R}$ is said to be *freely infinitely divisible* (or FID for short) if for each $n \in \{1,2,3,\dots\}$ there exists a probability measure μ_n such that $\mu = \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n-times}$.

2.3. **The Cauchy-Stieltjes Transform and Free Convolution.** The analytic approach to free convolution uses the Cauchy transform

$$G_\mu(z) = \int_{\mathbf{R}} \frac{1}{z-y} \mu(dy). \tag{2.1}$$

of a probability measure μ. It is analytic on the upper half plane $\mathbb{C}^+ = \{x+iy | s,t \in \mathbf{R}, y>0\}$ and takes values in the closed lower half plane $\mathbb{C}^- \cup \mathbb{R}$. The Cauchy transform has an inverse at a neighbourhood of infinity which has the form

$$G_\mu^{-1}(z) = \frac{1}{z} + R_\mu(z)$$

where $R_\mu(z)$ is analytic in a neighbourhood of zero and is called *R-transform.* Then free convolution is defined (see [38]) via the identity

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu. \tag{2.2}$$

The coefficients of the R-transform

$$R_X(z) = \sum_{n=0}^{\infty} K_{n+1}(X)\, z^n. \tag{2.3}$$

are called *free cumulants* of the random variable X.
The Cauchy transform is related to the moment generating function M_X as follows:

$$M_X(z) = \sum_{n=0}^{\infty} \tau(X^n)\, z^n = \frac{1}{z} G_X\left(\frac{1}{z}\right). \tag{2.4}$$

2.4. **Some probability distributions.** Let us now recall basic properties of some specific probability distributions which play prominent roles in the present paper.

2.4.1. *Wigner semicircular distribution.* A non-commutative random variable X is said to be free normal variable (i.e. have Wigner semicircular distribution) if the Cauchy-Stieltjes transform is given by the formula

$$G_\mu(z) = \frac{z - \sqrt{z^2-4}}{2}, \tag{2.5}$$

where $|z|$ is big enough, where the branch of the analytic square root should be determined by the condition that $\Im(z) > 0 \Rightarrow \Im(G_\mu(z)) \leqslant 0$ (see [30]). Equation (2.5) describes the family of distributions with mean zero and variance one (see [11, 30]). This measure has density

$$\frac{\sqrt{4-x^2}}{2\pi},$$

on $-2 \le x \le 2$. The Wigner semicircular distribution have cumulants $K_i = 0$ for $i > 2$.

2.4.2. *Free Poisson distribution.* A non-commutative random variable X is said to be free-Poisson variable if it has Marchenko-Pastur (or free-Poisson) distribution $\nu = \nu(\lambda, \alpha)$ defined by the formula

$$\nu = \max\{0,\, 1-\lambda\}\, \delta_0 + \tilde{\nu}, \tag{2.6}$$

where $\lambda \geq 0$ and the measure $\tilde{\nu}$, supported on the interval $(\alpha(1-\sqrt{\lambda})^2,\, \alpha(1+\sqrt{\lambda})^2)$, $\alpha > 0$ has the density (with respect to the Lebesgue measure)

$$\tilde{\nu}(dx) = \frac{1}{2\pi\alpha x}\sqrt{4\lambda\alpha^2 - (x-\alpha(1+\lambda))^2}\, dx.$$

The parameters λ and α are called the rate and the jump size, respectively. It is worth to note that a non-commutative variable with Marchenko-Pastur distribution arises also as a limit in law (in non-commutative sense) of variables with distributions $((1-\frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\delta_\alpha)^{\boxplus N}$ as $N \to \infty$, see [27]. Therefore, such variables are often called free-Poisson. It is easy to see that if X is free-Poisson, $\nu(\lambda, \alpha)$, then $K_n(X) = \alpha^n \lambda$. Therefore its R-transform has the form

$$R(z) = \frac{\lambda\alpha}{1-\alpha z}.$$

2.4.3. *Free chi-square distribution.* Let $X_1, \dots, X_n$ be free identically distributed random variables from the Wigner semicircular distribution with non-zero variance σ^2 and mean zero, and $\delta = \sum_{i=1}^n m_i^2$ $(m_i \in \mathbb{R})$. We call the distribution of the random variable $\sum_{i=1}^n (X_i + m_i)^2$ the free chi-square distribution with n degrees of freedom and noncentrality parameter δ, and we denote this distribution $\chi^2(n, \sigma, \delta)$ (a first version of this definition was introduced in [15]). In terms of R-transforms, a random variable Y has distribution $\chi^2(n, \sigma, \delta)$ if and only if

$$R_Y(z) = \frac{n\sigma^2}{1-\sigma^2 z} + \frac{\delta}{(1-2z)^2}. \tag{2.7}$$

If $\delta = 0$, the free chi-square distribution is called central, otherwise non-central and then we will write $\chi^2(n, \sigma)$ and from (2.7) we see that $\chi^2(n, \sigma)$ has the Marchenko-Pastur distribution $\nu(n, \sigma^2)$. Moreover, we will use the notation $\chi^2(n) := \chi^2(n, 1)$. It was shown in [15] that these distributions form a semigroup, namely $\chi^2(n_1, \sigma, \delta_1) \boxplus \chi^2(n_2, \sigma, \delta_2) = \chi^2(n_1 + n_2, \sigma, \delta_1 + \delta_2)$.

2.4.4. *Even elements.* We call an element $X \in \mathcal{A}$ even if all its odd moments vanish, i.e. $\tau(X^{2i+1}) = 0$ for all $i \geq 0$. It is immediately seen that the vanishing of all odd moments is equivalent to the vanishing of all odd cumulants, i.e., $K_{2i+1}(X) = 0$ and thus the even cumulants contain the complete information information about the distribution of an even element. The sequence $\alpha_n = K_{2n}(X)$ is called the *determining sequence* of X.

2.4.5. *Odd elements.* We call an element $X \in \mathcal{A}$ *odd* if $K_2(X) > 0$ and all its even free cumulants of order higher than two vanish, i.e. if $K_{2i}(X) = 0$ for all $i \geq 2$.
The basic example of such a law is Wigner's semicircular distribution. The classical analogue of odd elements only include the normal distribution because otherwise we could construct a normal random variable which is the sum of independent non-normal random variables (see below for the free case). This contradicts Cramér's decomposition theorem. However the free analogues of Marcinkiewicz' and Cramér's theorems fail. Bercovici and Voiculescu [3] showed that there exist probability distributions μ_ϵ with free cumulants $K_1(X) = 0$, $K_2(X) = 1$, $K_3(X) = \epsilon$ and $K_i(X) = 0$ for $i \geq 4$ if ϵ is small enough. This is an odd element and thus an explicit counterexample to the free analogue of Marcinkiewicz' theorem. To invalidate Cramér's theorem, take free copies X_1 and X_2 of random variables with distribution μ_ϵ, then the difference $X_1 - X_2$ is semicircular. Chistyakov and Götze [7] gave a detailed description of laws with finitely many free cumulants of arbitrary order. Thus an abundance of odd laws exists.

2.5. **The main result.** The main result of this paper is the following characterization of odd elements in terms of the sample variance. The proof of this theorem is given in Section 3.
The sample variance of a finite sequence of random variables X_i is the quadratic form

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{1}{n}\Big(1 - \frac{1}{n}\Big)\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\sum_{i,j=1,\ i\neq j}^{n} X_iX_j = \frac{1}{n^2}\sum_{1\leq i<j\leq n}(X_i - X_j)^2. \tag{2.8}$$

However in order to simplify notation in the present paper we chose to consider and call "sample variance" the rescaled quadratic form $Q_n = nS_n^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2$.
Our main result resolves the free analogue of χ^2-conjecture.

Theorem 2.1. *Let $X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa}$ be free copies of a random variable with finite non-zero variance σ^2. Then Q_n is distributed according to $\chi^2(n-1,\sigma)$ if and only if X is odd.*

Depending on the point of view it can be interpreted both as a positive and a negative solution. Taking into account the failure of Marcinkiewicz theorem this confirms the free analogue of the χ^2-conjecture in the broad sense.
If we suppose in addition that the distribution is even, then the above theorem gives a positive answer to the free analogue of Ruben's first theorem [28].

Proposition 2.2. *Let $X_1, X_2, \ldots, X_n$ be free identically distributed random variables with finite non-zero variance σ^2, and assume that the distribution of X_1 is symmetric. Then Q_n is distributed as $\chi^2(n-1,\sigma)$ if and only if X_1 has Wigner semicircular law.*

On the other hand, the free analogue of Ruben's second theorem [29] (see also [5]) does not hold:

Proposition 2.3. *Let $X_1, X_2, \ldots$ denote independently and identically distributed random variables with finite non-zero variance σ^2. Let m, n denote distinct integers not less than 2. Then for Q_n/σ^2 and Q_m/σ^2 to be distributed as $\chi^2(n-1)$ and $\chi^2(m-1)$, respectively, it is not necessary that X_1 is semicircular.*

Proof of Proposition 2.2. If Q_n is distributed as $\chi^2(n-1,\sigma)$ then X_1 is odd, but taking into account that X_1 is symmetric we have that its odd central moments vanish, and therefore its odd cumulants higher than the first vanish, so X_1 has Wigner semicircular distribution.
Proof of Proposition 2.3. Assume that $K_1(X_1) = 0$, $K_2(X_1) = \sigma^2$, $K_3(X_1) = \epsilon$ and $K_i(X_1) = 0$ for $i \geq 4$ where ϵ is small enough. By Theorem 2.1 we see that Q_n and Q_m have $\chi^2(n-1,\sigma)$ and $\chi^2(m-1,\sigma)$ distribution respectively.

□

Remark 2.4. In this paper we assume that the involved random variables are bounded, that is $X_i \in \mathcal{A}$, as was common practice for a long time. Recently however unbounded random variables, i.e., operators affiliated with the von Neumann algebra in question, came into the focus of research. This happened in particular in connection with certain characterization problems, see, e.g., [8, 10, 9]. It follows from the following result Chistyakov and Goetze that for the characterization problems pertinent to the present paper the question of boundedness is unessential.

Lemma 2.5 ([7, Lemma 3.10])**.** *Assume that $\mu = \mu_1 \boxplus \mu_2$, where μ has compact support. Then μ_1 and μ_2 have compact support as well.*

In terms of operators this means that if $X, Y \in \widetilde{\mathcal{A}}_{sa}$ are free random variables such that $X + Y \in \mathcal{A}_{sa}$, i.e., $X + Y$ is bounded, then $X, Y \in \mathcal{A}_{sa}$. Now we will show that Theorem 2.1 is true under weaker conditions.

Corollary 2.6. *Let $X_1, X_2, \ldots, X_n \in \widetilde{\mathcal{A}}_{sa}$ be selfadjoint free random variables and assume $Q_n = \frac{1}{n}\sum_{i<j}(X_i - X_j)^2$ is bounded. Then all X_i are bounded.*

Proof. Since X_i are self-adjoint and $nQ_n = \sum_{1\leq i\leq j\leq n}(X_i - X_j)^2$ is bounded, it follows that $(X_i - X_j)^2$ is bounded and hence also $X_i - X_j$. By Lemma 2.5 we deduce that all X_i are bounded. □

The proof of Ruben's theorem [28] heavily relies on the symmetry of random variables. Is it possible to drop the hypothesis that the random variables are symmetric? Golikova and Kruglov [13] give a partial answer to this question – instead of symmetry of X_1 they assume infinite divisibility. The following is a free version of their result which characterizes the classical normal law by the sample variance. We drop the assumption that X_i have the same distribution, because with this assumption the result follows directly from Theorem 2.1 (we cannot use the Bercovici-Pata bijection to prove it because it does not map classical chi-square to free chi-square distributions).

Proposition 2.7. *Let $X_1, X_2, \dots X_n$ denote free independent, infinitely divisible random variables with finite non-zero variance σ^2 with mean $\tau(X_1) = \tau(X_2) = \dots = \tau(X_n)$ and $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \dots = \mathrm{Var}(X_n) = 1$. Then if Q_n is distributed as free $\chi^2(n-1)$ if and only if $X_1, ..., X_n$ are identically distributed Wigner semicircular random variables.*

We conclude with a free version of a the following result of Kagan and Letac [18]: Fix an integer $n \geq 3$ and let $X_1, X_2, \dots, X_n$ be independent identically distributed random variables. Consider the linear subspace $E = \mathbf{1}^\perp$ of Euclidean space $\mathbb{R}^n$, i.e., the hyperplane $E = \{(a_1, a_2, \cdots, a_n) : a_1 + a_2 + \cdots + a_n = 0\}$. Then the following characterizations hold:

(i) If the distribution of the E-valued random variable

$$V = (X_1 - \overline{X}, \dots, X_n - \overline{X})$$

is invariant under all rotations of the Euclidean space E, then the X_i's are normally distributed.

(ii) If the distribution of the random variable

$$\sum_{i=1}^{n}(X_i - \overline{X} + a_i)^2$$

does not change as the real parameters a_i vary on a sphere (i.e., the euclidean length $\|a\|^2 = a_1^2 + a_2^2 + \cdots + a_n^2$ remains constant), then the X_i's are normally distributed.

A key ingredient of the proof of these classical results is played by *Marcinkiewicz' theorem.* As we discussed above, Marcinkiewicz' theorem has no analogue in free probability and we will use different methods to prove the following free version of [18]. This method also works in classical probability if we assume that all moment exists.

Proposition 2.8. *Let n be a fixed integer $n \geq 3$. Let $X_1, X_2, \dots, X_n$ be free identically distributed random variables.*

(1) If for all $a \in E \subset \mathbb{R}^n$ the distribution of the random variable

$$\sum_{i=1}^{n}(X_i - \overline{X} + a_i)^2$$

depends only on $\|a\|^2 = a_1^2 + a_2^2 + \cdots + a_n^2$, then the X_i's obey the semicircle law.

(2) If the distribution of the E-valued random variable

$$V = (X_1 - \overline{X}, \dots, X_n - \overline{X})$$

is invariant under all rotations of the Euclidean space E, then the X_i's obey the semicircle law.

2.6. **Noncrossing Partitions.** Let S be finite subset of $\mathbb{N}$. A partition of S is a set of mutually disjoint subsets (also called *blocks*) $B_1, B_2, \ldots, B_k \subseteq S$ whose union is S. Any partition π defines an equivalence relation on S, denotes $\sim_\pi$, such that the equivalence classes are the blocks π. That is, $i \sim_\pi j$ if i and j belong to the same block of π. A partition π is called *noncrossing* if different blocks do not interlace, i.e., there is no quadruple of elements $i < j < k < l$ such that $i \sim_\pi k$ and $j \sim_\pi l$ but $i \not\sim_\pi j$.
The set of non-crossing partitions of S is denoted by $NC(S)$, in the case where $S = [n] := \{1, \ldots, n\}$ we write $NC(n) := NC([n])$. $NC(n)$ is a poset under refinement order, where we say $\pi \leq \rho$ if every block of π is contained in a block of ρ. It turns out that $NC(n)$ is in fact a lattice, see [27, Lecture 9].
The maximal element of $NC(n)$ under this order is denoted by $\hat{1}_n$. It is the partition consisting of only one block. On the other hand the minimal element $\hat{0}_n$ is the unique partition where every block is a singleton.
Sometimes it is convenient to visualize partitions as diagrams, for example $\hat{1}_n = \sqcap\!\sqcap\cdots\sqcap$ and $\hat{0}_n = \mid\mid\cdots\mid$.

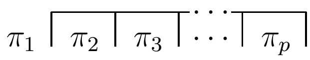
2.7. **Some Special Notations.** We will be concerned with certain special classes of noncrossing partitions. If n is even we denote by $NCE(n)$ the subset of even noncrossing partitions, where we say that a partition is even if all its blocks have even cardinality. Even more specific we denote by $NC_2(n)$ is the set of all noncrossing pair partitions, i.e., partitions where every block has size 2.
Two specific minimal pair partitions will play a particularly important role, namely $\hat{1}_2^r = \sqcap\,\sqcap\cdots\sqcap \in NC(2r)$, which is a kind of blow up of $\hat{1}_r$ and its shift $\nu_{0r} = \ulcorner\sqcap\,\sqcap\cdots\sqcap\urcorner \in NC(2r)$.
In the proof of Theorem 2.1 we will use telescoping argument and put a filtration on $NC(n)$ by avoiding certain blocks. For this purpose we introduce the following notation.
For a subset $B \subseteq \mathbb{N}$ let $NC^B(n) := \{\pi \in NC(n) : B \in \pi\}$, i.e., the collection of noncrossing partitions which contain B as a block. On the other hand, for a family $B_1, B_2, \ldots, B_m \subseteq \mathbb{N}$ of subsets let $NC_{B_1,\ldots,B_m}(S) := \{\pi \in NC(S) : \pi \cap \{B_1, \ldots, B_m\} = \emptyset\}$, i.e., the collection of noncrossing partition which do not contain any B_i as a block. Finally, combining the two notations we define $NC^B_{B_1,\ldots,B_m}(n) := NC^B(n) \cap NC_{B_1,\ldots,B_m}(n)$.

2.8. **Kreweras Complements.** Kreweras [20] discovered an interesting antiisomorphism of the lattice $NC(n)$, now called the *Kreweras complementation map*, of which we will need two variants. Given a noncrossing partition π of $\{1, 2, \ldots, n\}$, the *left Kreweras complement* $\overleftarrow{\pi}$ is the maximal noncrossing partition of the ordered set $\{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ such that $\pi \cup \overleftarrow{\pi}$ is a noncrossing partition of the interlaced set $\{\bar{1}, 1, \bar{2}, 2, \ldots, \bar{n}, n\}$. Similarly, the *right Kreweras complement* $\overrightarrow{\pi}$ is the maximal noncrossing partition of the ordered set $\{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ such that $\pi \cup \overrightarrow{\pi}$ is a noncrossing partition of the interlaced set $\{1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}\}$. It is then clear that $\overrightarrow{\circ}\overleftarrow{=}$id and it can be shown that

$$|\overrightarrow{\pi}| = |\overleftarrow{\pi}| = n + 1 - |\pi|. \tag{2.9}$$

Finally we define the *extended Kreweras complement* $\overleftrightarrow{\pi}$ to be the maximal noncrossing partition of the ordered set $\{\bar{0}, \bar{1}, \ldots, \bar{n}\}$ such that $\pi \cup \overleftrightarrow{\pi}$ is a noncrossing partition of the interlaced set $\{\bar{0}, 1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}\}$. The extended Kreweras complement is always irreducible, i.e., $\bar{0}$ and $\bar{n}$ are in the same block of $\overleftrightarrow{\pi}$. In fact it is obtained by joining $\bar{0}$ to the last block of $\overrightarrow{\pi}$, i.e., the block containing $\bar{n}$, or by joining $n + 1$ to the first block of $\overleftarrow{\pi}$. The following observation is useful for recursive proofs involving the Kreweras complement(s).

Lemma 2.9. *Let $\pi \in NC(n)$ and $B = \{j_1, j_2, \ldots, j_p = n\}$ be its last block. Let $\pi_1, \pi_2, \ldots, \pi_p$ be the restrictions of π to the maximal intervals of $\{1, 2, \ldots, n\} \setminus B$ as shown in the following picture:*



Then the left Kreweras complement of π *is the concatenation of the extended Kreweras complements of the subpartitions* π_j*:*

$$\overleftharpoon{\pi} = \widehat{\pi_1}\,\widehat{\pi_2}\cdots\widehat{\pi_p}.$$

2.9. **Free Cumulants.** Let $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ denote the non-commutative ring of polynomials in variables $X_1, \ldots, X_n \in \mathcal{A}$. The free cumulants are multilinear maps $K_r : \mathcal{A}^r \to \mathbb{C}$ defined implicitly by the relation (connecting them with mixed moments)

$$\tau(X_1X_2\ldots X_n) = \sum_{\pi\in NC(n)} K_\pi(X_1, X_2, \ldots, X_n), \tag{2.10}$$

where

$$K_\pi(X_1, X_2, \ldots, X_n) := \Pi_{B\in\pi} K_{|B|}(X_i : i \in B) \tag{2.11}$$

and $NC(n)$ is the set of all non-crossing partitions of $\{1, 2, \ldots, n\}$ (see [27]). Sometimes we will write $K_r(X) = K_r(X, \ldots, X)$.

Free cumulants provide the most important technical tool to investigate free random variables. This is due to the basic property of *vanishing of mixed cumulants*. By this we mean the fact that

$$K_r(X_1, X_2, \ldots, X_n) = 0$$

for any family of random variables $X_1, X_2, \ldots, X_n$ which can be partitioned into two free subsets. For free sequences this can be reformulated as follows. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of free random variables and $h : [r] \to \mathbb{N}$ a map. We denote by $\ker h$ the set partition which is induced by the equivalence relation

$$i \sim_{\ker h} j \Longleftrightarrow h(i) = h(j).$$

Using this notation, we have that

$$K_\pi(X_{h(1)}, X_{h(2)}, \ldots, X_{h(r)}) = 0 \text{ unless } \ker h \geq \pi. \tag{2.12}$$

Our main technical tool is the free version, due to Krawczyk and Speicher [19] (see also [27, Theorem 11.12]), of the classical formula of James/Leonov and Shiryaev [16, 23] which expresses cumulants of products in terms of individual cumulants.

Theorem 2.10. *Let* $r, n \in \mathbb{N}$ *and* $i_1 < i_2 < \cdots < i_r = n$ *be given and let*

$$\rho = \{(1, \ldots, i_1), \ldots, (i_{r-1} + 1, \ldots, i_r)\} \in NC(n)$$

be the induced interval partition. Consider now random variables $X_1, \ldots, X_n \in \mathcal{A}$*. Then the free cumulant of the products can be expanded as follows:*

$$K_r(X_1 \ldots X_{i_1}, \ldots, X_{i_{r-1}+1} \ldots X_n) = \sum_{\substack{\pi\in NC(n)\\ \pi\vee\rho=\hat{1}_n}} K_\pi(X_1, \ldots, X_n). \tag{2.13}$$

In the special case of products of pairs of free elements this yields the following formula for multiplicative free convolution.

Theorem 2.11 ([27, Theorem 14.4])**.** *Let* $\{X_1, X_2, \ldots, X_r\}$ *and* $\{Y_1, Y_2, \ldots, Y_r\}$ *be two mutually free sets of random variables, then*

$$K_r(X_1Y_1, X_2Y_2, \ldots, X_rY_r) = \sum_{\pi\in NC(r)} K_\pi(X_1, X_2, \ldots, X_r)\, K_{\overrightharpoon{\pi}}(Y_1, Y_2, \ldots, Y_r)$$

This motivates the following definition.

Definition 2.12 ([27, Ch. 17])**.** Let $f(z) = \sum_{n=1}^\infty a_n z^n$ and $g(z) = \sum_{n=1}^\infty b_n z^n$ be two formal power series. Their *boxed convolution* is defined as $f \boxstar g(z) = \sum_{n=1}^\infty c_n z^n$, where

$$c_n = \sum_{\pi\in NC(n)} a_\pi b_{\overrightharpoon{\pi}} = \sum_{\pi\in NC(n)} a_{\overleftharpoon{\pi}} b_\pi.$$

As examples, consider the univariate case of Theorem 2.11, which can be rewritten as $R_{XY}(z) = R_X \boxtimes R_Y(z)$, or [27, Proposition 11.25], which states that the R-transform of an even element X can be written as

$$R_{X^2}(z) = \alpha \boxtimes \zeta(z)$$

where $\alpha(z) = \sum_{n=1}^{\infty} K_{2n}(X) z^n$ is the determining series of X and $\zeta(z) = \sum_{n=1}^{\infty} z^n$ is the so-called *Zeta-series.* Combinatorially this means

$$K_r(X^2) = \sum_{\pi \in NC(r)} \alpha_\pi = \sum_{\pi \in NC(r)} \prod_{B \in \pi} K_{2|B|}(X). \tag{2.14}$$

The next result follows from [15, Proposition 2.2]; see Corollary 4.11 below for a generalization.

Proposition 2.13. *Let $X_1, X_2, \dots, X_n$ be free identically distributed Wigner semicircular random variables with mean zero and variance σ^2. Then the cumulants of Q_n are given as follows:*

$$K_r(Q_n) = (n-1)\,\sigma^{2r}. \tag{2.15}$$

The following lemma connects Theorem 2.10 with Definition 2.12 and is the key to the main result. Its proof is contained in the proof of Proposition 11.25 in the book [27].

Lemma 2.14. *Let $r \in \mathbb{N}$ and $\pi \in NCE(2r)$, then $\pi \vee \hat{1}_2^r = \hat{1}_{2r}$ if and only if $\pi \geq \nu_{0r}$, i.e., 1 and $2r$ lie in the same block of π and elements $2i$ and $2i+1$ also lie in the same block of π for $i \in [r-1]$. Consequently*

$$\{\pi : \pi \vee \hat{1}_2^r = \hat{1}_{2r}\} \cap NCE(2r) = [\nu_{0r}, \hat{1}_{2r}],$$

is a lattice isomorphic to $NC(r)$.

Corollary 2.15. *There is only one non-crossing pair partition π such that $\pi \vee \hat{1}_2^r = \hat{1}_{2r}$, namely $\nu_{0r} = \ulcorner\sqcap\,\sqcap\cdots\sqcap\urcorner = \{(1, 2r), (2, 3), \dots, (2r-2, 2r-1)\}$.*

Definition 2.16. Let $B_1, \dots, B_m$ be subsets of $\mathbb{N}$ and random variables $X_1, \dots, X_n \in \mathcal{A}$, be given. Then for an interval partition $\rho = \{(1, \dots, i_1), (i_1+1, \dots, i_2), \dots, (i_{r-1}+1, \dots, i_r)\}$ we define the partial cumulant functional

$$K^\rho_{B_1,\dots,B_m}(X_1, X_2, \dots, X_n) = \sum_{\substack{\pi \in NC_{B_1,\dots,B_m}(n) \\ \pi \vee \rho = \hat{1}_n}} K_\pi(X_1, \dots, X_n), \tag{2.16}$$

Usually we will abuse notation and abbreviate this expression as

$$K^r_{B_1,\dots,B_m}(X_1 \dots X_{i_1}, \dots, X_{i_{r-1}+1} \dots X_{i_r}) = \sum_{\substack{\pi \in NC_{B_1,\dots,B_m}(n) \\ \pi \vee \rho = \hat{1}_n}} K_\pi(X_1, \dots, X_n). \tag{2.17}$$

Lemma 2.17. *Let $P = P(X_1, X_2, \dots, X_n)$ be a polynomial of degree at most two in noncommuting variables $X_1, X_2, \dots, X_n$. Then $\tau(P(X_1, X_2, \dots, X_n)) = 0$ for every i.i.d. free family X_i if and only if*

$$\sum_{\sigma \in \mathfrak{S}_n} P(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(n)}) = 0.$$

Proof. Clearly by symmetry the second condition is stronger than the first condition.
In order to show that it is also necessary, we first note that by a simple scaling argument we may assume without loss of generality that the polynomial in consideration is homogeneous. Clearly such a polynomial cannot have a constant term and we start with a linear polynomial $P = \sum_{i=1}^n \alpha_i X_i$. By evaluating a distribution with nonzero first moment it follows that $\sum_{i=1}^n \alpha_i = 0$. But then we have

$$\sum_{\sigma \in \mathfrak{S}_n} P(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(n)}) = \sum_{i=1}^n \alpha_i (n-1)! \sum_{k=1}^n X_k = 0.$$

Let us now turn to a homogeneous polynomial of second order

$$P = \sum_{i,j=1}^{n} \alpha_{ij} X_i X_j.$$

Evaluating at a distribution with first moment μ_1 and second moment μ_2 we obtain

$$\sum_{i\neq j} \alpha_{ij}\mu_1^2 + \sum_i \alpha_{ii}\mu_2 = 0$$

and it follows that

$$\sum_{i\neq j} \alpha_{ij} = \sum_i \alpha_{ii} = 0.$$

Now consider the symmetrization

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_n} P(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(n)}) &= \sum_{i,j}\sum_{\sigma} X_{\sigma(i)}X_{\sigma(j)} \\ &= \sum_{i\neq j} \alpha_{ij}(n-2)! \sum_{k\neq l} X_k X_l + \sum_i \alpha_{ii}(n-1)! \sum_k X_k^2 \\ &= 0.\end{aligned}$$

□

Remark 2.18. (1) Our typical example of a centered linear statistic will be $X_i - \overline{X}$.
(2) The example $P = X_1X_2X_1 - X_1^2X_2$ shows that in the present formulation the lemma cannot be extended beyond degree 2.

In the following a polynomial $P(X_1, X_2, \dots, X_n)$ in noncommuting variables is called *symmetric* if it is invariant under permutations, i.e., $P(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(n)}) = P(X_1, X_2, \dots, X_n)$ for any permutation $\sigma \in \mathfrak{S}_n$. For a linear form $L = \sum_{i=1}^n \alpha_i X_i$ we denote the permuted form by $L_\sigma = \sum_{i=1}^n \alpha_i X_{\sigma(i)}$.

Lemma 2.19. *Let $X_1, X_2, \dots, X_n \in \mathcal{A}_{sa}$ be free identically distributed random variables, $L = \sum_{i=1}^n \alpha_i X_i$ a linear form such that $\tau(L) = 0$ and $P_j = P_j(X_1, X_2, \dots, X_n)$ symmetric polynomials for $j \in \{1, 2, \dots, r\} \setminus \{k\}$. Then*

$$K^r_{B_1,\dots,B_m}(P_1, \dots, P_{k-1}, L, P_{k+1}, \dots, P_r) = 0.$$

Proof. Let us first observe that for $i \neq j$ we have

$$K^r_{B_1,\dots,B_m}(P_1, \dots, P_{k-1}, X_i, P_{k+1}, \dots, P_r) = K^r_{B_1,\dots,B_m}(P_1, \dots, P_{k-1}, X_j, P_{k+1}, \dots, P_r),$$

for all $i, j \in [n]$, and $r \geq 1$. This follows from the argument that X_i are free i.i.d. and P_j are symmetric polynomials in the n variables $X_1, \dots, X_n$. It follows by multilinearity that

$$K^r_{B_1,\dots,B_m}(P_1, \dots, P_{k-1}, L, P_{k+1}, \dots, P_r) = K^r_{B_1,\dots,B_m}(P_1, \dots, P_{k-1}, L_\sigma, P_{k+1}, \dots, P_r),$$

for every permutation $\sigma \in \mathfrak{S}_n$ and taking the average, we have

$$K^r_{B_1,\dots,B_m}(P_1, \dots, P_{k-1}, L, P_{k+1}, \dots, P_r) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} K^r_{B_1,\dots,B_m}(P_1, \dots, P_{k-1}, L_\sigma, P_{k+1}, \dots, P_r) = 0,$$

again by multilinearity and taking into account Lemma 2.17.

□

Corollary 2.20. *Let $X_1, X_2, \dots, X_n \in \mathcal{A}_{sa}$ be free identically distributed random variables then*

$$K^r_{B_1,\dots,B_m}(Q_n, \dots, Q_n, X_i - \overline{X}, Q_n, \dots, Q_n) = 0.$$

3. Proof of the main theorem

Continuing Lemma 2.17 we establish a curious cancellation result for symmetrized squares of centered linear statistics. A similar phenomenon was observed by Nica and Speicher [26, Theorem 1.2] in the case of the free commutator. We postpone the investigation of a possible common pattern between these phenomena to future work.

Lemma 3.1. *Let $X_1, X_2, \dots, X_n$ be free identically distributed copies of a random variable X and $L = \sum_{i=1}^n \alpha_i X_i$ a linear form with $\tau(L) = 0$. Then the distribution of the quadratic statistic*

$$P = \sum_{\sigma \in \mathfrak{S}_n} L_\sigma^2,$$

does not depend on the odd cumulants of X.

Proof. We show by induction that the cumulants of P can be expressed in terms of the even cumulants of X. First we apply the product formula of Theorem 2.10 and obtain

$$\begin{aligned} K_r(P) &= \sum_{\sigma_1,\dots,\sigma_r \in \mathfrak{S}_n} K_r(L_{\sigma_1}^2, L_{\sigma_2}^2, \dots, L_{\sigma_r}^2) \\ &= \sum_{\sigma_1,\dots,\sigma_r \in \mathfrak{S}_n} \sum_{\substack{\pi \in NC(2r) \\ \pi \vee \hat{1}_2^r = \hat{1}_{2r}}} K_\pi(L_{\sigma_1}, L_{\sigma_1}, L_{\sigma_2}, L_{\sigma_2}, \dots, L_{\sigma_r}, L_{\sigma_r}) \\ &= \sum_{\substack{\pi \in NC(2r) \\ \pi \vee \hat{1}_2^r = \hat{1}_{2r}}} \tilde{K}_\pi(L) \end{aligned}$$

where for $\pi \in NC(2r)$ we write

$$\tilde{K}_\pi(L) = \sum_{\sigma_1,\sigma_2,\dots,\sigma_r \in \mathfrak{S}_n} K_\pi(L_{\sigma_1}, L_{\sigma_1}, L_{\sigma_2}, L_{\sigma_2}, \dots, L_{\sigma_r}, L_{\sigma_r}). \tag{3.1}$$

We claim that in this decomposition the contributions of non-even partitions cancel each other. To see this, we proceed by induction and use Lemma 2.19. Let $B_1, B_2, \dots, B_m$ be an enumeration of all odd subsets of $[2r]$, then we can split off the sum (3.1) the even part and decompose the rest in a "telescope" fashion as

$$K_r(P) = \sum_{\substack{\pi \in NCE(2r) \\ \pi \vee \hat{1}_2^r = \hat{1}_{2r}}} \tilde{K}_\pi(L) + \sum_{k=1}^m \sum_{\substack{\pi \in NC^{B_k}_{B_1,B_2,\dots,B_{k-1}}(2r) \\ \pi \vee \hat{1}_2^r = \hat{1}_{2r}}} \tilde{K}_\pi(L). \tag{3.2}$$

The last formula is obtained using the following decomposition

$$NC(2r) \setminus NCE(2r) = NC^{B_1}(2r) \cup NC^{B_2}_{B_1}(2r) \cup NC^{B_3}_{B_1,B_2}(2r) \cup \dots \cup NC^{B_m}_{B_1,\dots,B_{m-1}}(2r).$$

Directly from the definition we have $NC^{B_i}_{B_1,\dots,B_{i-1}}(2r) \cap NC^{B_j}_{B_1,\dots,B_{j-1}}(2r) = \emptyset$, for $i \neq j$, with the convention that $NC^{B_i}_{B_0}(2r) := NC^{B_1}(2r)$. We will show $\subseteq$ only because the opposite inclusion is obvious. Given $\pi \in NC(2r) \setminus NCE(2r)$, let k be the smallest index such that $B_k \in \pi$ then $\pi \in NC^{B_k}_{B_1,\dots,B_{k-1}}(2r)$ for the blocks $B_1, B_2, \dots, B_{k-1}$ do not appear in π.
It remains to show that each individual sum

$$\sum_{\substack{\pi \in NC^{B_k}_{B_1,B_2,\dots,B_{k-1}}(2r) \\ \pi \vee \hat{1}_2^r = \hat{1}_{2r}}} \tilde{K}_\pi(L) \tag{3.3}$$

vanishes. Every π in this sum contains the odd block B_k and splits the complement $[2r] \setminus B_k$ into intervals $I_1, I_2, \dots, I_l$, interpreted in a circular manner, see Fig. 1. Then at least one of these intervals must be odd. To simplify the discussion we may assume that either I_1 is odd and $2r \in B_k$ or I_l is odd and $1 \in B_k$; this may always be achieved by applying an even rotation, which does not change the values of the cumulants because of traciality. We are now in one of the situations depicted in Fig. 1. We concentrate on the first case, i.e.,

FIGURE 1. Two types of partitions with an odd block

$I_1 = \{1, 2, \dots, j_1 - 1\}$ and $B_k = \{j_1, j_2, \dots, j_l\}$ where j_1 is even, and $j_l = 2r$. Observe that every partition $\pi \in NC^{B_k}_{B_1,B_2,\dots,B_{k-1}}(2r)$ is the concatenation of some noncrossing partition $\pi' \in NC_{B_1,B_2,\dots,B_{k-1}}(j_1 - 1)$ and $\pi'' \in NC^{B_k}_{B_1,B_2,\dots,B_{k-1}}(\{j_1, \dots, 2r\})$. Now $\pi \vee \hat{1}^r_2 = \hat{1}_{2r}$ if and only if $\pi' \vee \sqcap\,\sqcap\,\cdots\,\sqcap\,| = \hat{1}_{j_1-1}$ and $\pi'' \vee |\,\sqcap\,\sqcap\,\cdots\,\sqcap = \hat{1}_{\{j_1,\dots,2r\}}$. Thus we may unfold (3.1) and factor the sum (3.3) to obtain

$$
\begin{aligned}
\sum_{\substack{\pi \in NC^{B_k}_{B_1,B_2,\dots,B_{k-1}}(2r) \\ \pi \vee \hat{1}^r_2 = \hat{1}_{2r}}} \tilde{K}_\pi(L) &= \sum_{\substack{\pi' \in NC_{B_1,B_2,\dots,B_{k-1}}(j_1-1) \\ \pi' \vee \sqcap\,\sqcap\,\cdots\,\sqcap\,| = \hat{1}_{j_1-1}}} \tilde{K}_{\pi'}(L) \sum_{\substack{\pi'' \in NC^{B_k}_{B_1,B_2,\dots,B_{k-1}}(\{j_1,\dots,2r\}) \\ \pi'' \vee |\,\sqcap\,\sqcap\,\cdots\,\sqcap = \hat{1}_{\{j_1,\dots,2r\}}}} \tilde{K}_{\pi''}(L) \\
&= \sum_{\sigma_1,\sigma_2,\dots,\sigma_r \in \mathfrak{S}_n} \sum_{\substack{\pi' \in NC_{B_1,B_2,\dots,B_{k-1}}(j_1-1) \\ \pi' \vee \sqcap\,\sqcap\,\cdots\,\sqcap\,| = \hat{1}_{j_1-1}}} K_{\pi'}(L_{\sigma_1}, L_{\sigma_1}, \dots, L_{\sigma_{j'_1-1}}, L_{\sigma_{j'_1-1}}, L_{\sigma_{j'_1}}) \\
&\qquad \times \sum_{\substack{\pi'' \in NC^{B_k}_{B_1,B_2,\dots,B_{k-1}}(\{j_1,\dots,2r\}) \\ \pi'' \vee |\,\sqcap\,\sqcap\,\cdots\,\sqcap = \hat{1}_{\{j_1,\dots,2r\}}}} K_{\pi''}(L_{\sigma_{j'_1}}, L_{\sigma_{j'_1+1}}, \dots, L_{\sigma_r}, L_{\sigma_r}) \\
&= \sum_{\sigma_{j'_1},\sigma_{j'_1+1},\dots,\sigma_r \in \mathfrak{S}_n} K^{j'_1}_{B_1,B_2,\dots,B_{k-1}}(P, P, \dots, P, L_{\sigma_{j'_1}}) \\
&\qquad \times \sum_{\substack{\pi'' \in NC^{B_k}_{B_1,B_2,\dots,B_{k-1}}(\{j_1,\dots,2r\}) \\ \pi'' \vee |\,\sqcap\,\sqcap\,\cdots\,\sqcap = \hat{1}_{\{j_1,\dots,2r\}}}} K_{\pi''}(L_{\sigma_{j'_1}}, L_{\sigma_{j'_1+1}}, \dots, L_{\sigma_r}, L_{\sigma_r}).
\end{aligned}
$$

And by Lemma 2.19 the factor

$$K^{j'_1}_{B_1,B_2,\dots,B_{k-1}}(P, P, \dots, P, L_\sigma),$$

vanishes for every σ. Here we use the notation $j'_i = j_{\lceil i/2 \rceil}$ where $\lceil \cdot \rceil$ is the ceiling function which rounds up to the nearest integer. □

Remark 3.2. (1) Note that in the case of sample variance we have to assume identical distribution of the involved random variables for the cancellation phenomenon to take place; in the case of the free commutator this requirement is not necessary.

(2) The argument put forward in the previous proof is not valid in classical probability except in the case where B_k is an interval block. For example if $r = 3$ and $B = \{1, 3, 6\}$ then the block B alone ensures that $\pi \vee \sqcap\,\sqcap\,\sqcap = \sqcap\!\sqcap\!\sqcap\!\sqcap\!\sqcap$ and thus

$$\sum_{\substack{\pi \in \mathcal{P}^B(6) \\ \pi \vee \sqcap\,\sqcap\,\sqcap = \sqcap\sqcap\sqcap\sqcap\sqcap}} K_\pi(T_1, T_1, T_2, T_2, T_3, T_3) = K_3(T_1, T_2, T_3)\, \tau(T_1 T_2 T_3),$$

where the sum runs over all set partitions.

Proof of Theorem 2.1. Let $X_1, \dots, X_n$ be free copies of a fixed random variable X. We apply Lemma 3.1 for $L = X_1 - \overline{X}$ and $P = \sum_{\sigma \in \mathfrak{S}_n} L^2_\sigma = (n-1)!\, Q_n$ to conclude from (3.2) that

$$K_r(Q_n) = \left(\frac{1}{(n-1)!} \right)^r \sum_{\substack{\pi \in NCE(2r) \\ \pi \vee \hat{1}^r_2 = \hat{1}_{2r}}} \tilde{K}_\pi(L).$$

Now the fact that X_i are free identically distributed implies that in the sum (3.1) every term either vanishes or is a multiple of $K_\pi(X)$. Therefore we may write

$$K_r(Q_n) = \sum_{\pi \in NCE_0(2r)} c_n(\pi)\, K_\pi(X) \tag{3.4}$$

where $NCE_0(2r) = \{\pi \in NCE(2r) \mid \pi \vee \hat{1}_2^r = \hat{1}_{2r}\}$ and $c_n(\pi) \in \mathbb{R}$. We will derive an explicit formula for these coefficients in Section 4 below. Using identity (3.4) we will show that all even cumulants of higher order vanish, i.e., $K_{2i}(X) = 0$ for $i \geq 2$. First let us compute the parameters $c_n(\pi)$ in the extreme cases $\pi = \nu_{0r}$ and $\pi = \hat{1}_{2r}$.

For $\nu_{0r} = \ulcorner\sqcap\sqcap\cdots\sqcap\urcorner$ only the second cumulant contributes to $c_n(\nu_{0r})$ and the value of the latter does not change if we replace X with a centered semicircular variable of variance σ^2. In this case Corollary 2.15 implies that

$$K_r(Q_n) = c_n(\ulcorner\sqcap\sqcap\cdots\sqcap\urcorner)\, \sigma^{2r},$$

and from Proposition 2.13 we infer that $c_n(\ulcorner\sqcap\sqcap\cdots\sqcap\urcorner) = n - 1$.

To compute the value of $c_n(\hat{1}_{2r})$ it is convenient to switch to tensor notation and to identify the multilinear cumulant functional $K_r : \mathcal{A}^n \to \mathbb{C}$ with its linear extension $K_r : \mathcal{A}^{\otimes n} \to \mathbb{C}$. Let us now assume without loss of generality that $\sigma = 1$. We have to evaluate

$$\begin{aligned}
\tilde{K}_{\hat{1}_{2r}}(L) &= \sum_{i_1, i_2, \ldots, i_r = 1}^{n} K_{2r}(X_{i_1} - \overline{X}, X_{i_1} - \overline{X}, X_{i_2} - \overline{X}, X_{i_2} - \overline{X}, \ldots, X_{i_r} - \overline{X}, X_{i_r} - \overline{X}) \\
&= K_{2r}\Bigg(\bigg(\sum_{i=1}^{n} (X_i - \overline{X}) \otimes (X_i - \overline{X})\bigg)^{\otimes r}\Bigg) \\
&= K_{2r}\Bigg(\bigg(\sum_{i=1}^{n} X_i \otimes X_i - n\overline{X} \otimes \overline{X}\bigg)^{\otimes r}\Bigg).
\end{aligned}$$

Expanding this power yields cumulants of the form

$$\begin{aligned}
K_{2r}\Bigg(\bigg(\sum_{i=1}^{n} X_i \otimes X_i\bigg)^{\otimes k} \otimes \Big(-n\overline{X} \otimes \overline{X}\Big)^{\otimes (r-k)}\Bigg) &= nK_{2r}\Bigg(\Big(X_1 \otimes X_1\Big)^{\otimes k} \otimes \Big(-\frac{1}{n} X_1 \otimes X_1\Big)^{\otimes (r-k)}\Bigg) \\
&= n\left(-\frac{1}{n}\right)^{r-k} K_{2r}(X),
\end{aligned}$$

and in total

$$\tilde{K}_{\hat{1}_{2r}}(L) = n \sum_{k=0}^{r} \binom{r}{k} \left(-\frac{1}{n}\right)^{r-k} K_{2r}(X) = n\left(1 - \frac{1}{n}\right)^{r} K_{2r}(X).$$

Next, to evaluate even cumulants, equate the r-th cumulants of Q_n and $\chi^2(n-1)$, i.e.,

$$K_r(Q_n) = K_r(\chi^2(n-1)) = n - 1.$$

Denote $NCE_0'(2r) = NCE_0(2r) \setminus \{\ulcorner\sqcap\sqcap\cdots\sqcap\urcorner, \hat{1}_{2r}\}$, then we have

$$\begin{aligned}
n - 1 &= \sum_{\pi \in NCE_0(2r)} c_n(\pi)\, K_\pi(X) \\
&= \sum_{\pi \in NCE_0'(2r)} c_n(\pi)\, K_\pi(X) + \sum_{\substack{\pi \in NC_2(2r) \\ \pi \vee \hat{1}_2^r = \hat{1}_{2r}}} c_n(\pi)\, K_\pi(X) + c_n(\hat{1}_{2r})\, K_{2r}(X) \\
&= \sum_{\pi \in NCE_0'(2r)} c_n(\pi)\, K_\pi(X) + c_n(\ulcorner\sqcap\sqcap\cdots\sqcap\urcorner) + \frac{(n-1)^r}{n^{r-1}} K_{2r}(X) \\
&= \sum_{\pi \in NCE_0'(2r)} c_n(\pi)\, K_\pi(X) + n - 1 + \frac{(n-1)^r}{n^{r-1}} K_{2r}(X).
\end{aligned} \tag{3.5}$$

This yields

$$\sum_{\pi\in NCE_0'(2r)} c_n(\pi)\, K_\pi(X) + \frac{(n-1)^r}{n^{r-1}}\, K_{2r}(X) = 0. \tag{3.6}$$

and the blocks of any $\pi \in NCE_0'(2r)$ have size strictly smaller than $2r$, it follows by induction that $K_{2r}(X) = 0$ for $r \geq 2$.
Conversely, suppose that X_i's are odd, then from Lemma 3.1 we get

$$\begin{aligned} K_r(Q_n) &= \sum_{\pi\in NCE_0(2r)} c_n(\pi)\, K_\pi(X) = \sum_{\substack{\pi\in NC_2(2r)\\ \pi\vee\hat{1}_2^r=\hat{1}_{2r}}} c_n(\pi)\, K_\pi(X) \\ &= c_n(\sqcap\sqcap\cdots\sqcap) = n-1. \end{aligned}$$

□

Proof of Proposition 2.7. Recall that as a consequence of the free Lévy-Khinchin formula (see for example [27, Theorem 13.16]) the random variable X_i is infinitely divisible if and only if

$$K_{n+2}(X_i) = \int_{\mathbb{R}} x^n d\rho_i(x),$$

for some finite measure $\rho_i(x)$ on $\mathbb{R}$. For the semicircular distribution the measure is $\rho = \delta_0$ and it suffices to show that $\int x^2 d\rho_i(x) = K_4(X_i) = 0$. Now if $\tau(Y_1) = \tau(Y_2) = \tau(Y_3) = \tau(Y_4) = 0$ then the product formula from Theorem 2.10 implies

$$K_2(Y_1Y_2, Y_3Y_4) = K_2(Y_1, Y_4)\, K_2(Y_2, Y_3) + K_4(Y_1, Y_2, Y_3, Y_4). \tag{3.7}$$

We will apply this to Q_n, so let us first compute the cumulants which will appear after evaluation of (3.7). By assumption $K_2(X_i) = 1$ for all i and therefore the covariances are

$$K_2(X_i - \overline{X}, X_j - \overline{X}) = \begin{cases} \frac{n-1}{n} & \text{if } i = j, \\ -\frac{1}{n} & \text{if } i \neq j. \end{cases}$$

It remains to consider cumulants of order 4. First,

$$\begin{aligned} \sum_{i=1}^n K_4(X_i - \overline{X}) &= \sum_{i=1}^n \left(\left(1 - \frac{1}{n}\right)^4 K_4(X_i) + \sum_{l\neq i} \left(-\frac{1}{n}\right)^4 K_4(X_l) \right) \\ &= \left(\left(1 - \frac{1}{n}\right)^4 + \frac{n-1}{n^4} \right) \sum_{i=1}^n K_4(X_i), \end{aligned}$$

second,

$$\begin{aligned} &\sum_{\substack{i,j=1\\ i\neq j}}^n K_4(X_i - \overline{X}, X_i - \overline{X}, X_j - \overline{X}, X_j - \overline{X}) \\ &\qquad = \sum_{\substack{i,j=1\\ i\neq j}}^n \left(\left(1 - \frac{1}{n}\right)^2 \left(-\frac{1}{n}\right)^2 (K_4(X_i) + K_4(X_j)) + \sum_{l\neq i,j} \left(-\frac{1}{n}\right)^4 K_4(X_l) \right) \\ &\qquad = \left(2(n-1) \left(1 - \frac{1}{n}\right)^2 \left(\frac{1}{n}\right)^2 + \frac{(n-1)(n-2)}{n^4} \right) \sum_{i=1}^n K_4(X_i), \end{aligned}$$

and thus

$$\sum_{i,j=1}^n K_4(X_i - \overline{X}, X_i - \overline{X}, X_j - \overline{X}, X_j - \overline{X}) = \frac{(n-1)^2}{n^2} \sum_{i=1}^n K_4(X_i).$$

Using these formulas we now proceed to (3.7) and obtain

$$\begin{aligned} n-1 &= K_2(Q_n, Q_n) \\ &= \sum_{i,j=1}^{n} K_2((X_i-\overline{X})^2, (X_i-\overline{X})^2) \\ &= \sum_{i,j=1}^{n} K_2(X_i-\overline{X}, X_j-\overline{X})\, K_2(X_j-\overline{X}, X_i-\overline{X}) \\ &\qquad + \sum_{i,j=1}^{n} K_4(X_i-\overline{X}, X_i-\overline{X}, X_j-\overline{X}, X_j-\overline{X}) \\ &= \sum_{i=1}^{n} [K_2((X_i-\overline{X}),(X_i-\overline{X}))]^2 + \sum_{i=1,j=1,i\neq j}^{n} [K_2(X_i-\overline{X}, X_j-\overline{X})]^2 \\ &\quad + \sum_{i=1}^{n} K_4(X_i-\overline{X}) + \sum_{i=1,j=1,i\neq j}^{n} K_4(X_i-\overline{X}, X_i-\overline{X}, X_j-\overline{X}, X_j-\overline{X}) \\ &= \frac{(n-1)^2}{n} + \frac{n-1}{n} + \frac{(n-1)^2}{n^2}\sum_{i=1}^{n} K_4(X_i). \end{aligned}$$

So we see that $\sum_{i=1}^n K_4(X_i) = \sum_{i=1}^n \int_{\mathbb{R}} x^2 d\rho_i(x) = 0$ and thus $\rho_i(x) = \delta_0(x)$. Note that the above measure is the free Lévy measure of the semicircle distribution with mean zero, and variance one.

□

Proof of Proposition 2.8. Part (1). We write $a = \|a\|\theta = \|a\|(\theta_1, \ldots, \theta_n)$ where θ belongs to the unit sphere of E, i.e., $\sum \theta_i = 0$. Thus for $\|a\| > 0$, $a_i = \|a\|\theta_i$ and $r \geq 2$ we have

$$\begin{aligned} \frac{K_r(\sum_{i=1}^n (X_i - \overline{X} + a_i)^2)}{\|a\|^r} &= K_r\Big(\sum_{i=1}^n (X_i-\overline{X})^2/\|a\| - \sum_{i=1}^n 2(X_i-\overline{X})\,\theta_i + \|a\|\Big) \\ &= K_r\Big(\sum_{i=1}^n (X_i-\overline{X})^2/\|a\| - \sum_{i=1}^n 2(X_i-\overline{X})\,\theta_i\Big) \\ &= K_r\Big(\sum_{i=1}^n (X_i-\overline{X})^2/\|a\| - \sum_{i=1}^n 2X_i\theta_i\Big). \end{aligned} \tag{3.8}$$

By the hypothesis the left hand side of (3.8) does not depend on θ, and thus the limit on the right hand side

$$\lim_{\|a\|\to+\infty} \frac{K_r(\sum_{i=1}^n (X_i - \overline{X} + a_i)^2)}{\|a\|^r} = K_r\Big(-\sum_{i=1}^n 2X_i\theta_i\Big),$$

does not depend on θ either. Now freeness implies that

$$S_r(\theta_1, \ldots, \theta_n) := K_r\Big(\sum_{i=1}^n X_i\theta_i\Big) = \sum_{i=1}^n \theta_i^r K_r(X_i) = \Big(\sum_{i=1}^n \theta_i^r\Big) K_r, \tag{3.9}$$

is a constant function on the unit sphere of the space E. Thus we see $S_2(\theta_1, \ldots, \theta_n) = K_2$ and S_r for $r \geq 3$ is constant function on the unit sphere of the space E if and only $K_r = 0$ for $r \geq 3$. We now show part (2) of Proposition 2.8. It's easy to observe that for $\theta \in E$ we have

$$K_r\Big(\sum_{i=1}^n (X_i-\overline{X})\,\theta_i\Big) = \sum_{i=1}^n \theta_i^r K_r(X_i) = \Big(\sum_{i=1}^n \theta_i^r\Big) K_r, \tag{3.10}$$

is constant function on the unit sphere of the space E and from the above consideration we get the statement.

□

Remark 3.3. The preceding proof is also valid in the general setting of [21].

4. R-cyclic matrices and infinite divisibility of quadratic forms

In this section we show how the sample variance can be combined with the concept of R-cyclicity and give a recipe for computing the coefficients $c_n(\pi)$ in equation (3.4). In particular we also show that sample variance preserves free infinite divisibility.

4.1. **R-cyclic matrices and the distribution of sample variance.** The concept of R-cyclicity was introduced by Nica, Shlyakhtenko and Speicher [24]. Our aim is now to exhibit its relation to the sample variance and other quadratic forms. We show that the theory of R-cyclic matrices can by used to compute the distribution of the sample variance and give a formula for the cumulants of the sample variance in terms of the even cumulants, which generalizes and unifies two types of results, namely cumulants of squares of even elements [27, Proposition 11.25] and cumulants of quadratic forms in gaussian random variables [21, Proposition 4.4].
Here we consider matrices over a non-commutative probability space. Let $(\mathcal{A},\tau)$ be a non-commutative probability space, and let n be a positive integer. The algebra $M_n(\mathcal{A})$ of $n\times n$ matrices over $\mathcal{A}$ is a noncommutative probability space with canonical expectation functional

$$\tau_n(A)=\frac{1}{n}\sum_{i=1}^n\tau(a_{i,i}),$$

where $A=[a_{i,j}]_{i,j=1}^n$ is a matrix in $M_n(\mathcal{A})$. Then $(M_n(\mathcal{A}),\tau_n)$ is itself a non-commutative probability space. The definition of R-cyclicity is in terms of the joint R-transform of the entries of the matrix: one requires that only the cyclic non-crossing cumulants of the entries are allowed to be different from 0, see Definition 4.1 below. Equivalently, it was shown in [24, Theorem 8.2] that R-cyclicity is equivalent to the fact that A is free from $M_n(\mathbb{C})$ with amalgamation over the algebra $\mathcal{D}_n$ of scalar diagonal matrices with respect to the conditional expectation

$$\begin{aligned}E^{\mathcal{D}}&:M_n(\mathcal{A})\to M_n(\mathbb{C})\\ A&\mapsto\sum_{i=1}^n E_i\tau^{(n)}(A)E_i,\end{aligned}\tag{4.1}$$

where by E_i we denote the projection matrix onto the i-th unit vector and $\tau^{(n)}(A)_{ij}=\tau(a_{i,j})$ is the entry-wise trace.

Definition 4.1. Let $(M_n(\mathcal{A}),\tau_n)$ and n be as above, then a matrix $\boldsymbol{X}=[X_{i,j}]_{i,j=1}^n\in M_n(\mathcal{A})$. is said to be *$R$-cyclic* if for every $r\geq1$ and for every choice of indices $1\leq i_1,j_1,\dots,i_r,j_r\leq n$ the cumulant

$$K_r(X_{i_1,j_1},X_{i_2,j_2},\dots,X_{i_r,j_r})=0,$$

unless the indices are cyclic in the sense that $j_1=i_2,j_2=i_3,\dots,j_{r-1}=i_r,j_r=i_1$. In this case the formal noncommutative power series

$$f_{\boldsymbol{X}}(z_1,z_2,\dots,z_n)=\sum_{r=1}^\infty\sum_{i_1,i_2,\dots,i_r=1}^n K_r(X_{i_r,i_1},X_{i_1,i_2},\dots,X_{i_{r-1},i_r})\,z_{i_1}z_{i_2}\cdots z_{i_r},\tag{4.2}$$

is called the *determining series* of the entries of $\boldsymbol{X}$.

The concept of R-cyclicity generalizes the concept of R-diagonality [27, Ex. 20.5] in the sense that X is R-diagonal if and only if the matrix $\left[\begin{smallmatrix}0&X\\X^*&0\end{smallmatrix}\right]$ is R-cyclic.

Lemma 4.2. *For scalar matrices $A\in M_n(\mathbb{C})$ we have*

(i)

$$\sum_{i=1}^n E_iA_1E_iA_2\cdots E_iA_rE_i=E^{\mathcal{D}}(A_1)E^{\mathcal{D}}(A_2)\cdots E^{\mathcal{D}}(A_r).$$

(ii) Let $\pi\in NC(r)$, then

$$\sum_{\ker\underline{i}\geq\pi}E^{\mathcal{D}}(A_1E_{i_1}A_2E_{i_2}\cdots A_rE_{i_r}A_{r+1})=E^{\mathcal{D}}_{\hat{\pi}}(A_1,A_2,\dots,A_{r+1}).$$

Proof. Part (i) follows immediately from the expansion

$$\sum_{k=1}^{n} E_k A_1 E_k A_2 \cdots A_r E_k = \sum_{k=1}^{n} E_k a_{k,k}^{(1)} a_{k,k}^{(2)} \cdots a_{k,k}^{(r)}.$$

To see part (ii) we single out the last block of π (i.e., the block containing r, see Lemma 2.9), say $B = \{j_1 < j_2 < \cdots < j_p = r\}$, and group the remaining blocks into subpartitions, empty partitions allowed, say $\pi_1 \in NC([1, j_1 - 1])$, $\pi_2 \in NC([j_1 + 1, j_2 - 1]), \dots, \pi_p \in NC([j_{p-1} + 1, j_p - 1])$. Then we have

$$E^{\mathcal{D}}\Bigl(\sum_{\ker \underline{i} \geq \pi} (A_1 E_{i_1} A_2 E_{i_2} \cdots E_{i_r} A_{r+1})\Bigr) = E^{\mathcal{D}}\Bigl(\sum_{i} A'_1 E_i A'_2 E_i \dots A'_p E_i A_{r+1}\Bigr),$$

where

$$A'_k = \sum_{\ker \underline{i} \geq \pi_k} A_{j_{k-1}+1} E_{i_1} A_{j_{k-1}+2} E_{i_2} \cdots A_{j_k}.$$

By part (i) this is

$$E^{\mathcal{D}}(A'_1 E^{\mathcal{D}}(A'_2) \dots E^{\mathcal{D}}(A'_p) A_{r+1}),$$

and by induction this is

$$E^{\mathcal{D}}(A'_1 E^{\mathcal{D}}_{\widehat{\pi_2}}(A_{j_1+1}, \dots, A_{j_2}) \cdots E^{\mathcal{D}}_{\widehat{\pi_p}}(A_{j_{p-1}+1}, \dots, A_r) A_{r+1}) = E^{\mathcal{D}}_{\widehat{\pi}}(A_1, A_2, \dots, A_{r+1}),$$

where we used Lemma 2.9. ☐

Proposition 4.3. *Let $X_1, X_2, \dots, X_n \in \mathcal{A}$ be a free family of even random variables and $A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathbb{C})$ a scalar matrix. Then the Schur product matrix*

$$\mathbf{Z} = A \circ \boldsymbol{X} = [a_{i,j} X_i X_j]_{i,j=1}^n = \begin{bmatrix} a_{1,1}X_1^2 & a_{1,2}X_1X_2 & \dots & a_{1,n}X_1X_n \\ a_{2,1}X_2X_1 & a_{2,2}X_2^2 & \dots & a_{2,n}X_2X_n \\ \dots & \dots & \dots & \dots \\ a_{n,1}X_nX_1 & a_{n,2}X_nX_2 & \dots & a_{n,n}X_n^2 \end{bmatrix}, \tag{4.3}$$

is R-cyclic.

Proof. We make use of the product formula of Theorem 2.10 and Lemma 2.14 to compute

$$\begin{aligned} K_r(X_{i_1}X_{i_2}, X_{i_3}X_{i_4}, \dots, X_{i_{2r-1}}X_{i_{2r}}) &= \sum_{\substack{\pi \in NCE(2r) \\ \pi \vee \hat{1}_2^r = \hat{1}_{2r}}} K_\pi(X_{i_1}, X_{i_2}, \dots, X_{i_{2r}}) \\ &= \sum_{\substack{\pi \in NCE(2r) \\ \pi \geq \nu_{0r}}} K_\pi(X_{i_1}, X_{i_2}, \dots, X_{i_{2r}}) \end{aligned} \tag{4.4}$$

and by (2.12) these mixed cumulants vanish unless $\ker \underline{i} \geq \nu_{0r}$, i.e., unless $i_1 = i_{2r}$ and $i_{2j} = i_{2j+1}$ for all j, which exactly means R-cyclicity. It is easy to see that the same holds for $Z_{i,j} = a_{i,j} X_i X_j$. ☐

Remark 4.4. In some sense Proposition 4.3 is a generalization of the fact [27, Theorem 20.6] that the product of two free even selfadjoint elements is R-diagonal. This fact is indeed a consequence if we put $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ in the preceding theorem. In fact it was shown in [14] that every R-diagonal element can be written as a product of two free even selfadjoint elements. It is an interesting question what would be a natural factorization of R-cyclic matrices. While it is necessary for a matrix to be R-cyclic that its entries form R-diagonal pairs, example [27, Ex. 20.6] shows that the representation (4.3) in the preceding proposition does not cover all R-cyclic matrices.

Proposition 4.5. *Let $X_1, X_2, \dots, X_n \in \mathcal{A}$ be a free family of even random variables, $\boldsymbol{X} = [X_i X_j]_{i,j=1}^n$ and $A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathbb{C})$ a scalar matrix.*

(i) *The determining series of the entries of the R-cyclic matrix $A \circ \boldsymbol{X}$ and the R-transform of the quadratic form $T_n = \sum_{i,j}^n a_{i,j} X_i X_j$ are related by*

$$f_{A\circ \boldsymbol{X}}(z,\dots,z) = \mathcal{R}_{T_n}(z), \tag{4.5}$$

where $\mathcal{R}_{T_n}(z) = zR_{T_n}(z)$.

(ii) *The cumulants of T_n are given by*

$$\begin{aligned} &K_r(T_n) \\ &= \sum_{i_1,\dots,i_r\in[n]} \operatorname{Tr}(AE_{i_1}AE_{i_2}\dots AE_{i_r}) \sum_{\substack{\pi\in NCE(2r)\\ \pi\vee\hat{1}_2^r=\hat{1}_{2r}}} K_\pi(X_{i_r},X_{i_1},X_{i_1},X_{i_2},\dots,X_{i_{r-1}},X_{i_r}). \end{aligned} \tag{4.6}$$

(iii) *If we assume in addition that X_i are identically distributed the previous formula simplifies to the following convolution-like expression*

$$K_r(T_n) = \sum_{\pi\in NC(r)} \operatorname{Tr}(E_\pi^{\mathcal{D}}(A)) \prod_{B\in\pi} K_{2|B|}(X). \tag{4.7}$$

Proof. From the definition of T_n we see that

$$\begin{aligned} K_r(T_n) &= \sum_{i_1,i_2,\dots,i_{2r}\in[n]} K_r(Z_{i_1,i_2},Z_{i_3,i_4},\dots,Z_{i_{2r-1},i_{2r}}) \\ &= \sum_{\substack{i_1,i_2,\dots,i_{2r}\in[n]\\ \ker \underline{i}\geq\nu_{0r}}} \sum_{\substack{\pi\in NCE(2r)\\ \pi\geq\nu_{0r}}} a_{i_1,i_2}a_{i_3,i_4}\cdots a_{i_{2r-1},i_{2r}} K_\pi(X_{i_1},X_{i_2},\dots,X_{i_{2r}}), \end{aligned}$$

where we used (4.4). Having eliminated the zero contributions we can apply Lemma 2.14 in the reverse direction and obtain

$$\begin{aligned} &= \sum_{\substack{i_1,i_2,\dots,i_{2r}\in[n]\\ \ker \underline{i}\geq\nu_{0r}}} K_r(Z_{i_1,i_2},Z_{i_3,i_4},\dots,Z_{i_{2r-1},i_{2r}}) \\ &= \sum_{i_1,i_2,\dots,i_r} K_r(Z_{i_r,i_1},Z_{i_1,i_2},\dots,Z_{i_{r-1},i_r}), \end{aligned}$$

which after comparison with (4.2) yields (4.5). We now expand further and obtain

$$\begin{aligned} &= \sum_{i_1,i_2,\dots,i_r\in[n]} a_{i_r,i_1}a_{i_1,i_2}\cdots a_{i_{r-1},i_r} K_r(X_{i_r}X_{i_1},X_{i_1}X_{i_2},\dots,X_{i_{r-1}}X_{i_r}) \\ &= \sum_{i_1,\dots,i_r\in[n]} \operatorname{Tr}(AE_{i_1}AE_{i_2}\dots AE_{i_r}) \sum_{\substack{\pi\in NCE(2r)\\ \pi\vee\hat{1}_2^r=\hat{1}_{2r}}} K_\pi(X_{i_r},X_{i_1},X_{i_1},X_{i_2},\dots,X_{i_{r-1}},X_{i_r}), \end{aligned}$$

which yields (4.6). Now denoting by $\hat{\pi}$ the image of $\pi\in NC(r)$ under the bijection introduced in Lemma 2.14 we can rewrite this as

$$= \sum_{\pi\in NC(r)} \left(\sum_{\ker \underline{i}\geq\pi} \operatorname{Tr}(AE_{i_1}AE_{i_2}\dots AE_{i_r}) \right) K_{\hat{\pi}}(X).$$

Finally we infer (4.7) from Lemma 4.2.

□

Remark 4.6. It was observed in [24, Rem. 4.1] that R-cyclicity is preserved under Hadamard products with constant matrices. Moreover inspecting the preceding proof one can easily see the that for an arbitrary R-cyclic matrix $\boldsymbol{X} = [X_{i,j}]$ and any scalar matrix $A = [a_{i,j}]$ the determining series of the Hadamard product $A\circ\boldsymbol{X} = [a_{i,j}X_{i,j}]$ is given by

$$f_{A\circ\boldsymbol{X}}(z_1,z_2,\dots,z_n) = \operatorname{Tr}(f_{\boldsymbol{X}}(AE_1\otimes z_1, AE_2\otimes z_2,\dots,AE_n\otimes z_n)).$$

In fact we have proved the following slightly more general statement.

Theorem 4.7. *Let X_i be free copies of an even random variable X, $\boldsymbol{X} = [X_iX_j]_{i,j=1}^n$ be a matrix of products as above and let $A_1, A_2, \dots, A_r \in M_n(\mathbb{C})$ be arbitrary scalar matrices. Then $(A_1 \circ \boldsymbol{X}, A_2 \circ \boldsymbol{X}, \dots, A_r \circ \boldsymbol{X})$ is an R-cyclic family and the joint cumulant of $T_k = \sum_{ij} a_{ij}^{(k)} X_iX_j$ is*

$$K_r(T_1, T_2, \dots, T_r) = \sum_{\pi \in NC(r)} \operatorname{Tr}(E_\pi^{\mathcal{D}}(A_1, A_2, \dots, A_r)) \prod_{B \in \pi} K_{2|B|}(X).$$

It was shown in [24, Section 8] that R-cyclicity of a matrix is equivalent to freeness from the algebra of constant matrices $M_n(\mathbb{C})$ with amalgamation over the commutative subalgebra $\mathcal{D}_n$ of constant diagonal matrices. Moreover, the cyclic scalar cumulants can be interpreted as entries of the $\mathcal{D}_n$-valued cumulants as follows.

Proposition 4.8 ([24, Theorem 7.2]). *Let $(\boldsymbol{X}_i) \subseteq M_n(\mathcal{A})$ be an R-cyclic family over some noncommutative probability space $(\mathcal{A}, \tau)$ and denote by $K_r^{\mathcal{D}}$ the operator valued cumulant functionals with respect to the conditional expecation* (4.1)*. Then for any $\Lambda_1, \Lambda_2, \dots, \Lambda_{r-1} \in \mathcal{D}_n$ we have*

$$\begin{aligned} &K_r^{\mathcal{D}}(\boldsymbol{X}_1\Lambda_1, \boldsymbol{X}_2\Lambda_2, \dots, \boldsymbol{X}_{r-1}\Lambda_{r-1}, \boldsymbol{X}_r) \\ &\qquad = \sum_{i_1, i_2, \dots, i_r = 1}^n \lambda_{i_1}^{(1)} \lambda_{i_2}^{(2)} \cdots \lambda_{i_{r-1}}^{(r-1)} K_r(X_{i_r,i_1}^{(1)}, X_{i_1,i_2}^{(2)}, \dots, X_{i_{r-2},i_{r-1}}^{(r-1)}, X_{i_{r-1},i_r}^{(r)}) E_{i_r}. \end{aligned}$$

In our context this leads to an operator valued boxed convolution in the sense of [32, Definition 2.1.6] as follows.

Proposition 4.9. *Let $X_1, X_2, \dots, X_n \in \mathcal{A}$ be free even copies of a random variable X and let $\boldsymbol{X} = [X_iX_j]_{i,j=1}^n$. Then for any scalar matrices $A_1, A_2, \dots, A_r \in M_n(\mathbb{C})$ and $\Lambda_1, \Lambda_2, \dots, \Lambda_{r-1} \in \mathcal{D}_n$ we have*

$$\begin{aligned} &K_r^{\mathcal{D}}(A_1 \circ \boldsymbol{X}\Lambda_1, A_2 \circ \boldsymbol{X}_2\Lambda_2, \dots, A_{r-1} \circ \boldsymbol{X}_{r-1}\Lambda_{r-1}, A_r \circ \boldsymbol{X}_r) \\ &\qquad = \sum_{\pi \in NC(r)} E_\pi^{\mathcal{D}}(A_1\Lambda_1, A_2\Lambda_2, \dots, A_{r-1}\Lambda_{r-1}, A_r) \prod_{B \in \pi} K_{2|B|}(X). \end{aligned}$$

Proof. We use Proposition 4.8 and expand

$$\begin{aligned} &K_r^{\mathcal{D}}(A_1 \circ \boldsymbol{X}\Lambda_1, A_2 \circ \boldsymbol{X}_2\Lambda_2, \dots, A_{r-1} \circ \boldsymbol{X}_{r-1}\Lambda_{r-1}, A_r \circ \boldsymbol{X}_r) \\ &\quad = \sum_{i_1, i_2, \dots, i_r = 1}^n a_{i_r,i_1}^{(1)} \lambda_{i_1}^{(1)} a_{i_1,i_2}^{(2)} \lambda_{i_2}^{(2)} \cdots a_{i_{r-1},i_r}^{(r)} K_r(X_{i_r}X_{i_1}, X_{i_1}X_{i_2}, \dots, X_{i_{r-1}}X_{i_r}) E_{i_r} \\ &\quad = \sum_{i_1, i_2, \dots, i_r = 1}^n a_{i_r,i_1}^{(1)} \lambda_{i_1}^{(1)} a_{i_1,i_2}^{(2)} \lambda_{i_2}^{(2)} \cdots a_{i_{r-1},i_r}^{(r)} \sum_{\substack{\pi \in NC(2r) \\ \pi \geq \nu_{0r}}} K_\pi(X_{i_r}, X_{i_1}, X_{i_1}, X_{i_2}, \dots, X_{i_{r-1}}, X_{i_r}) E_{i_r} \\ &\quad = \sum_{\substack{\pi \in NC(2r) \\ \pi \geq \nu_{0r}}} \sum_{\ker \underline{i} \geq \pi} a_{i_r,i_1}^{(1)} \lambda_{i_1}^{(1)} a_{i_1,i_2}^{(2)} \lambda_{i_2}^{(2)} \cdots a_{i_{r-1},i_r}^{(r)} E_{i_r} K_\pi(X) \\ &\quad = \sum_{\pi \in NC(r)} E_\pi^{\mathcal{D}}(A_1\Lambda_1, A_2\Lambda_2, \dots, A_{r-1}\Lambda_{r-1}, A_r) K_{\hat{\pi}}(X), \end{aligned}$$

where $\hat{\pi}$ is defined in the proof of Proposition 4.5. □

Remark 4.10. In fact it is easy to see that the matrix $\Xi = \operatorname{diag}(X_1, X_2, \dots, X_n)$ is free from $M_n(\mathbb{C})$ with amalgamation over $\mathcal{D}_n$ as well [24, Example 2.3]. We have shown above that $A \circ \boldsymbol{X} = \Xi A \Xi$ has the same property although A has not.

As a final corollary we obtain the following formula for the cumulants of the sample variance.

Corollary 4.11. *Let $X_1, X_2, \dots, X_n$ be free copies of a random variable X and $Q_n = nS_n^2$ the rescaled sample variance defined in (2.8). Let $\tilde{X}$ be the symmetrization of X, i.e., a formal random variable with even distribution and cumulants $K_{2r}(\tilde{X}) = K_{2r}(X)$. Then*

$$K_r(Q_n) = (n-1)K_r(Z^2), \tag{4.8}$$

where $Z = \sqrt{\frac{n}{n-1}}P\tilde{X}P$ is the free compression of the symmetrization $\tilde{X}$ of X by a projection P of trace $\tau(P) = \frac{n-1}{n}$.

Proof. By Lemma 3.1 the distribution of Q_n does not change if we drop the odd cumulants and replace X by its symmetrization $\tilde{X}$. The symmetrization $\tilde{X}$ being even, it follows from Proposition 4.5 that the information about the distribution of the sample variance is contained in the R-cyclic matrix $A \circ \tilde{\boldsymbol{X}} = [a_{ij}\tilde{X}_i\tilde{X}_j]_{i,j=1}^n$, where $A = I - \frac{1}{n}\mathbf{1}$. This matrix is idempotent with $E^{\mathcal{D}}(A) = (1-1/n)I$ and therefore $E_\pi^{\mathcal{D}}(A) = (1-\frac{1}{n})^{|\pi|}I$ for every $\pi \in NC(r)$. We insert this into (4.7) and the cumulants of Q_n evaluate to

$$K_r(Q_n) = n \sum_{\pi \in NC(r)} \left(1 - \frac{1}{n}\right)^{|\bar{\pi}|} \prod_{B\in\pi} K_{2|B|}(\tilde{X})$$

This in turn by (2.9) is equal to

$$= n\left(1-\frac{1}{n}\right)^{r+1} \sum_{\pi\in NC(r)} \prod_{B\in\pi} \frac{n}{n-1} K_{2|B|}(\tilde{X}).$$

$$= (n-1) \sum_{\pi\in NC(r)} \prod_{B\in\pi} \frac{n}{n-1} K_{2|B|}\left(\sqrt{1-\frac{1}{n}}\tilde{X}\right).$$

In view of (2.14) this is the same as the cumulant $K_r(Z^2)$ where Z is an even random variable with cumulants

$$K_r(Z) = \frac{n}{n-1} K_r\left(\sqrt{1-\frac{1}{n}}\tilde{X}\right).$$

Such a random variable can be modeled as a free compression

$$Z = \frac{n}{n-1} P \sqrt{1-\frac{1}{n}} \tilde{X} P = \sqrt{\frac{n}{n-1}} P\tilde{X}P,$$

with $\tau(P) = \frac{n-1}{n}$, see [27, Corollary 14.13]. ☐

Remark 4.12. In the paper [25] of Nica and Speicher cited above, it was shown that for every probability measure μ there is a convolution semigroup $\{\mu^{\boxplus t} \mid t \geq 1\}$. Denote $\psi(\mu) = \inf\{t \mid \mu^{\boxplus t} \text{ exists}\}$. This can be seen as some kind of "measure of non-infinite divisibility" in the sense that μ is $\boxplus$-infinitely divisible if and only if $\psi(\mu) = 0$. It is related to the $\boxplus$-divisibility indicator $\phi(\mu)$ of [2] by the inequality $\psi(\mu) \leq 1 - \phi(\mu)$. If $\tilde{X}$ exists, the preceding proof shows that $\psi(Z) \leq \frac{n-1}{n}\psi(\tilde{X})$ and in particular, if $\tilde{X}$ is $\boxplus$-infinitely divisible, then so is Z. It then follows from [1, Theorem 6.1] that Z^2 is infinitely divisible as well and consequently also Q_n. However if X is not infinite divisible, the symmetrization $\tilde{X}$ constructed in Corollary 4.11 in general cannot be realized as an operator, see [26, Remark 12 (2)].

We show in the final section that any quadratic form in free even random variables preserves infinite divisibility.

4.2. **Preservation of free infinite divisibility.** It is shown in [1] that the free commutator of $\boxplus$-infinitely divisible random variables is also $\boxplus$-infinitely divisible and the authors ask whether there are other noncommutative polynomials which preserve free infinite divisibility. We show here that for self-adjoint operators this is the case for any quadratic form in free random variables whose distribution does not depend on the odd cumulants of the original distribution. This includes the free commutator and free sample variance. In the proof below we will use compound free Poisson distributions μ with rate λ and jump distribution ν which is the unique probability distribution with free cumulants $K_n(\mu) = \lambda m_n(\nu)$. Compound free Poisson distributions are $\boxplus$-infinitely divisible, and moreover, any $\boxplus$-infinitely divisible probability measure is a weak limit distribution of a sequence of compound free Poisson random variables, see [1, Proposition A.2].

Proposition 4.13. *Let $X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa}$ be a free family of even $\boxplus$-infinitely divisible random variables. Let $A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathbb{C})$ be a selfadjoint matrix, then the distribution of the quadratic form $T_n = \sum_{i,j}^n a_{i,j} X_i X_j$ is also $\boxplus$-infinitely divisible.*

Proof. Suppose first that each X_i is a symmetric compound free Poisson variable with rate λ_i and jump distribution ν_i. Let Y_i be a free random variable, with compound free Poisson distribution of rate λ_i and jump distribution ν_i^2, i.e., with cumulants given by $K_r(Y_i) = K_{2r}(X_i)$. Using the equation (4.6) we have

$$K_r(T_n) = \sum_{i_1,\ldots,i_r \in [n]} \mathrm{Tr}(AE_{i_1} AE_{i_2} \ldots AE_{i_r}) \sum_{\substack{\pi \in NCE(2r) \\ \pi \vee \hat{1}_2^r = \hat{1}_{2r}}} K_\pi(X_{i_r}, X_{i_1}, X_{i_1}, X_{i_2}, \ldots, X_{i_{r-1}}, X_{i_r}),$$

under the bijection introduced in Lemma 2.14 and $K_r(Y_i) = K_{2r}(X_i)$ we get

$$\begin{aligned} &= \sum_{i_1,\ldots,i_r \in [n]} \mathrm{Tr}(AE_{i_1} AE_{i_2} \ldots AE_{i_r}) \sum_{\substack{\pi \in NC(r) \\ \pi \leq \ker \underline{i}}} K_\pi(Y_{i_1}, Y_{i_2}, \ldots, Y_{i_r}) \\ &= \sum_{i_1,\ldots,i_r \in [n]} \mathrm{Tr}(AE_{i_1} AE_{i_2} \ldots AE_{i_r}) \sum_{\pi \in NC(r)} K_\pi(Y_{i_1}, Y_{i_2}, \ldots, Y_{i_r}) \\ &= \sum_{i_1,\ldots,i_r \in [n]} \mathrm{Tr}(AE_{i_1} AE_{i_2} \ldots AE_{i_r}) \, \tau\Big(\prod_{j=1}^r Y_{i_j}\Big) = n \times \mathrm{Tr}_n \otimes \tau \left[\Big(\sum_{i=1}^n AE_i \otimes Y_i \Big)^r \right]. \end{aligned}$$

Hence the cumulant sequence of T_n is the moment sequence of $\sum_{i=1}^n AE_i \otimes Y_i$ in the probability space $M_n(\mathbb{C}) \otimes \mathcal{A}$, with state $\mathrm{Tr}_n \otimes \tau$.
Suppose now that X_i has a more general symmetric distribution μ_i. Then the argument of the proof of Proposition A.2. in [1] shows that μ_i can be approximated by symmetric compound free Poissons, say $\mu_i = \lim_{k\to\infty} \mu_{i,k}$. It follows from the above argument that the distribution T_n can be approximated by freely infinitely divisible distributions and since $ID(\boxplus)$ is closed under convergence in distribution, T_n is freely infinitely divisible as well. □

Putting together Lemma 3.1 and Proposition 4.13 we obtain the following corollary.

Corollary 4.14. *Let $X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa}$ be a free family of even $\boxplus$-infinitely divisible random variables. Let P be a selfadjoint symmectric polynomial of degree 2 in noncommuting variables such that the distribution of the random variable $Y = P(X_1, X_2, \ldots, X_n)$ does not depend on the odd cumulants. Then the distribution of Y is $\boxplus$-infinitely divisible as well. In particular, the commutator $i(X_1X_2 - X_2X_1)$ of two $\boxplus$-infinitely divisible random variables is $\boxplus$-infinitely divisible and the same is true of the sample variance of a free identically distributed family of $\boxplus$-infinitely divisible random variables.*

5. Concluding Remarks

In the present paper we have shown that the sample variance shares the following properties with the free commutator:

(1) Odd cumulants do not contribute to the distribution.
(2) Infinite divisibility is preserved.

This phenomenon raises the following problems and conjectures, some of which will be investigated in forthcoming papers.

Problem 5.1. Characterize the class of selfadjoint polynomials $P \in \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle$ in noncommuting variables $X_1, X_2, \dots, X_n$ with the property that the distribution of $P(X_1, \dots, X_n)$ does not depend on the odd cumulants of X whenever $X_1, X_2, \dots, X_n$ are free copies of a fixed random variable X.

Conjecture 5.2. *Whenever a homogeneous polynomial P has the properties described in Problem 5.1 and $X_1, X_2, \dots, X_n$ are free copies of a $\boxplus$-infinitely divisible random variable X, then $P(X_1, \dots, X_n)$ is $\boxplus$-infinitely divisible as well.*

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(Wiktor Ejsmont) Department of Mathematics and Cybernetics Wrocław University of Economics, ul. Komandorska 118/120, 53-345 Wrocław, Poland
E-mail address: wiktor.ejsmont@gmail.com

(Franz Lehner) Department of Discrete Mathematics TU Graz Steyrergasse 30, 8010 Graz, Austria
E-mail address: lehner@math.tugraz.at