

# Two associative operads of packed words

Samuele Giraudo<sup>\*1</sup> and Yannic Vargas<sup>+2</sup>

<sup>1</sup> *Université du Québec à Montréal, LACIM, Pavillon Président-Kennedy, 201 Avenue du Président-Kennedy, Montréal, H2X 3Y7, Canada.*

<sup>2</sup> *Institute of Discrete Mathematics, Technische Universität Graz, Austria, Graz, Austria.*

**Abstract.** The associative operad is a central structure in operad theory, defined on the linear span of the set of permutations. We build two analogs of the associative operad on the linear span of the set of packed words which turn out to be set-theoretical. By seeing a packed word as a surjective map between two finite sets, our first operad is graded by the cardinality of the domain and the second one, by the cardinality of the codomain. In the same way as the associative operad of permutations contains as quotients the duplicial and interstice operads, we derive similar structures for our operads of packed words. We propose also an analogue of Dynkin idempotent of Zie algebras in this context of operads of packed words.

**Keywords:** Operad; Associative operad; Permutation; Treelike structure; Packed word; Dynkin idempotent.

## Introduction

The associative operad  $\text{As}$  is an important algebraic structure which plays a central role in the theory of operads. This operad is defined on the linear span of the set of permutations and its partial composition consists of inserting a permutation into another one, interpreted as permutation matrices. The first reason justifying the importance of  $\text{As}$  is that it intervenes in a crucial way in the description of the axioms of symmetric operads. A second reason, shown by Aguiar and Livernet [2], relates to its richness from a combinatorial point of view. Indeed,  $\text{As}$  admits a basis defined using the left weak order on permutations which possesses the nice property that the partial composition of two elements of this basis is a sum of an interval of this partial order. Furthermore, This operad enjoys many interesting properties since it contains not only the Lie operad as a suboperad, but also, as quotients, the duplicial operad of binary trees [4] and the interstice operad on two generators of binary words [5, 6].

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<sup>\*</sup>[giraudo.samuele@uqam.ca](mailto:giraudo.samuele@uqam.ca). This research has been partially supported by the projects CARPLO (ANR-20-CE40-0007) and ALCOHOL (ANR-19-CE40-0006) of the Agence nationale de la recherche.

<sup>+</sup>[yvargaslozada@tugraz.at](mailto:yvargaslozada@tugraz.at). This research has been supported by Austrian Science Fund FWF, grant I 5788 PAGCAP.

These three operads on permutations, binary trees, and binary words form a hierarchy very similar to a well-known hierarchy of combinatorial Hopf algebras involving the same spaces of combinatorial objects, namely, the Malvenuto-Reutenauer Hopf algebra of permutations [17, 7], the Loday-Ronco Hopf algebra of binary trees [15], and the noncommutative symmetric functions Hopf algebra [8]. Interestingly enough, there is a generalization of the Malvenuto-Reutenauer Hopf algebra on the linear span of packed words. This Hopf algebra has been introduced by Hivert [11] when he considered a notion of word quasi-symmetric functions. This construction is natural since while permutations are bijections, packed words are surjections. In this context, the analog of the Loday-Ronco Hopf algebra involves Schröder trees [19] and the analog of the noncommutative symmetric functions Hopf algebra involves ternary words [19]. The starting point of the present work is to explore whether such a natural generalization of  $As$  exists and if it leads to a similar hierarchy of operads.

Our main contribution consists of the introduction of two different generalizations of  $As$  on the linear span of packed words. By seeing a packed word as its matrix, we obtain a right version  $PAs^{\rightarrow}$  consisting of inserting a packed word matrix at a given position into another one, and a left version  $PAs^{\leftarrow}$  consisting of inserting several copies of a packed word matrix onto given values. These operads also differ in the way the arity of a packed word is defined: in  $PAs^{\rightarrow}$  (resp.  $PAs^{\leftarrow}$ ), the arity of a packed word is the cardinality of its domain (resp. codomain). As they are not isomorphic as graded spaces,  $PAs^{\rightarrow}$  and  $PAs^{\leftarrow}$  are not isomorphic as operads. Moreover,  $PAs^{\leftarrow}$  is a symmetric operad while  $PAs^{\rightarrow}$  is not. Besides, the operad  $PAs^{\leftarrow}$  is not combinatorial in the sense that it admits infinitely many elements of any given arity  $n \geq 1$ . Nevertheless, despite this fact, this operad is rich from a combinatorial point of view since it admits several quotients using well-known equivalence relations on packed words, like the sylvester [12], hypoplactic [14], or Baxter [9] congruences.

This work is presented as follows. Section 1 contains fundamental notions about the main combinatorial objects and operads appearing in this work. In Section 2, we introduce and construct the operads  $PAs^{\rightarrow}$  and  $PAs^{\leftarrow}$  and present their first properties. Section 3 is devoted to the study of some of the quotients of these two operads of packed words which involve other families of combinatorial objects. Section 4 contains the construction of an analogue of the classical Dynkin idempotents for Zie algebras, introduced in [1]. Finally, Section 5 contains several open questions as well as avenues for future research in continuation of this work.

*General notations and conventions.* For any integer  $i$ ,  $[i]$  denotes the set  $\{1, \dots, i\}$ . For any set  $A$ ,  $A^*$  is the set of words on  $A$ . For any  $w \in A^*$ ,  $\ell(w)$  is the length of  $w$ , and for any  $i \in [\ell(w)]$ ,  $w(i)$  is the  $i$ -th letter of  $w$ . The only word of length 0 is the empty word  $\epsilon$ . For any  $1 \leq i \leq j \leq \ell(w)$ ,  $w(i, j)$  is the word  $w(i) \dots w(j)$ . Given two words  $w$  and  $w'$ , the concatenation of  $w$  and  $w'$  is denoted by  $ww'$  or by  $w.w'$ .

# 1 Preliminaries

## 1.1 Packed words and related objects

Let  $\mathbb{P}$  be the set  $\mathbb{N} \setminus \{0\}$ . For any  $u \in \mathbb{P}^*$  and  $\alpha, \beta \in \mathbb{N}$ , let  $\uparrow_\alpha^\beta(u)$  be the word obtained by incrementing by  $\alpha$  the letters of  $u$  which are greater than  $\beta$ . For instance,  $\uparrow_2^4(124546) = 124748$ . Given  $w \in \mathbb{P}^*$ , the *alphabet* of  $w$  is the set  $\text{Alph}(w)$  formed by all different letters appearing in  $w$ . An *inversion* of a word  $w \in \mathbb{P}^*$  is a pair  $(i, j)$  such that  $1 \leq i < j \leq \ell(w)$  and  $w(i) > w(j)$ . The *set of inversions* of  $w$  is denoted by  $\text{Inv}(w)$  and the *number of inversions* of  $w$  is  $\text{inv}(w) := |\text{Inv}(w)|$ .

The *standardization map*  $\text{st} : \mathbb{P}^* \rightarrow \mathfrak{S}$  is the function sending a word  $w$  of length  $n$  to a permutation  $\text{st}(w) \in \mathfrak{S}_n$ , obtained by iteratively scanning  $w$  from left to right, and labelling  $1, 2, 3, \dots$  the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. For instance, if  $x < y < z$  are letters in  $\mathbb{P}^*$ , then  $\text{st}(yyxzzxzyx) = 451782963 \in \mathfrak{S}_9$ . The standardization of a word  $w$  is the unique permutation of  $\mathfrak{S}_n$  preserving the relative order of  $w$ ; that is,  $w$  and  $\text{st}(w)$  have the same inversions (see [13]).

A *packed word* is a word  $w \in \mathbb{P}^*$  satisfying  $i - 1 \in \text{Alph}(w)$  whenever  $i \in \text{Alph}(w)$ , for every  $i \geq 2$ . This implies that the alphabet of a packed word  $w$  is such that  $\text{Alph}(w) = [k]$ , for some  $k \geq 0$ . Also, if  $n$  is the length of  $w$ , the definition of packed words necessarily implies that  $k \leq n$ . For every  $k, n \geq 0$ , let  $\mathfrak{P}_n[k]$  be the set of all packed words of alphabet  $[k]$  and length  $n$ . For every  $n \geq 0$ ,  $\mathfrak{S}_n := \mathfrak{P}_n[n]$  is the set of *permutations* of the set  $[n]$ .

A *Schröder tree* is a planar rooted tree where each internal node has at least two children. For every  $k, n \geq 0$ , let  $\mathfrak{T}_n[k]$  be the set of all Schröder trees with  $n + 1$  leaves and  $k$  internal nodes. Observe that  $\mathfrak{B}_n := \mathfrak{T}_n[n]$  is the set of *binary trees* with  $n + 1$  leaves (or equivalently, binary trees with  $n$  internal nodes), for every  $n \geq 0$ .

Given  $k, n \geq 0$  and  $\mathfrak{X}_k[n] \in \{\mathfrak{P}_k[n], \mathfrak{T}_k[n]\}$ , we put

$$\mathfrak{X}_n := \bigcup_{0 \leq k \leq n} \mathfrak{X}_k[n], \quad \mathfrak{X}[k] := \bigcup_{n \geq k} \mathfrak{X}_n[k], \quad \text{and} \quad \mathfrak{X} := \bigcup_{n \geq 0} \mathfrak{X}_n. \quad (1.1)$$

Here,  $\mathfrak{X}_n$  and  $\mathfrak{X}[k]$  correspond to the sets of our given combinatorial objects classified by *length* and *alphabet*, whenever  $\mathfrak{X} \in \{\mathfrak{P}, \mathfrak{T}\}$  represent packed words or Schröder trees, respectively.

Recall that the symmetric group acts on the vector space of non-commutative polynomials  $\mathbb{K}\langle X \rangle$ , for any countable set  $X$ , where  $\mathbb{K}$  is any field of characteristic zero. When  $X = \mathbb{P}$ , this action restricts to packed words: if  $w \in \mathfrak{P}_n[k]$  and  $\sigma \in \mathfrak{S}_n$ , then

$$w \cdot \sigma := w(\sigma(1)) \cdots w(\sigma(n)) \in \mathfrak{P}_n[k]. \quad (1.2)$$

Indeed, if  $w$  is a packed word, any permutation of its letters leads to a word satisfying the condition on the definition of packed words.

For every packed word  $w$ , let  $\chi(w)$  be the unique weakly increasing word obtained after rearrangement of  $w$ . We call  $\chi(w)$  the *composition type* of  $w$ . In particular,  $\chi(w)$  is again a packed word of same alphabet and length than  $w$ . Every weakly increasing packed word  $w \in \mathfrak{P}_n[k]$  can be encoded by the integer composition  $c := c_1c_2\dots c_k$  of  $n$  such that every  $i$  appears  $c_i$  times in  $w$ . Equivalently, through the classical bijection between subsets of  $[n-1]$  and binary strings of length  $n-1$ , the composition type  $\chi(w)$  of a packed word of length  $n$  can be also described as a word of length  $n-1$  on the alphabet  $\{1,2\}$ . For instance, if  $u = 2311223$  and  $v = 221$ , then

$$\chi(u) = 1122233 \leftrightarrow -+--+- \leftrightarrow 121121 \text{ and } \chi(v) = 122 \leftrightarrow +- \leftrightarrow 21. \quad (1.3)$$

The following lemma, due to Hivert, relates a packed word with its composition type.

**Lemma 1.1** (Hivert, [13]). *For any  $w \in \mathfrak{P}_n[k]$ ,  $w = \chi(w) \cdot st(w)$ . Moreover,  $st(w)$  is the smallest permutation in  $\mathfrak{S}_n$  for the right weak order satisfying the above property.*

This result implies that every packed word  $w$  is encoded by its composition type and its standardization: the composition type tells us how many times each letter appears in  $w$ , while  $st(w)$  encodes the relative order of the appearance of the letters in  $w$ .

## 1.2 Associative operad and quotients

We follow the usual notations about symmetric unital operads [16] (called simply *operads* here). Here are four important examples of (nonsymmetric) operads.

- The *right associative operad* is the symmetric operad  $As^\rightarrow$  (also written  $As$ ) wherein for any  $n \geq 1$ ,  $As^\rightarrow(n)$  is the linear span of the set  $\mathfrak{S}_n$ . The set  $\{E_\sigma : \sigma \in \mathfrak{S}_n, n \geq 1\}$  is a basis of  $As^\rightarrow$ . Given  $\alpha \in \mathfrak{S}_n$ ,  $\beta \in \mathfrak{S}_m$  and  $i \in [n]$ , let

$$B_i(\alpha, \beta) := \uparrow_{m-1}^{\alpha(i)}(\alpha(1, i-1)) \cdot \uparrow_{\alpha(i)-1}^0(\beta) \cdot \uparrow_{m-1}^{\alpha(i)-1}(\alpha(i+1, n)). \quad (1.4)$$

For instance,  $B_5(3612457, 231) = 381256479$ . The **red** labels reflect the **red** permutation 231 inserted into the **blue** permutation 3612457 onto the letter 4 at the 5-th position. The permutation  $B_i(\alpha, \beta)$  is sometimes called the *block permutation* associated to  $\alpha$  and  $\beta$  (see [16]). The partial composition of  $As^\rightarrow$  satisfies  $E_\sigma \circ_i E_\nu = E_{B_i(\sigma, \nu)}$  and the action of the symmetric group  $\mathfrak{S}_n$  satisfies  $E_\sigma \cdot \nu = E_{\sigma \circ \nu}$  where  $\circ$  is the composition of permutations.

- The *left associative operad* is the symmetric operad  $As^\leftarrow$  defined on the same space as the one of  $As^\rightarrow$ , seen on the same E-basis. The partial composition of  $As^\leftarrow$  satisfies  $E_\sigma \circ_i E_\nu = E_\pi$  where  $\pi$  is the permutation obtained by replacing in  $\uparrow_{m-1}^i(\sigma)$  the letter  $i$  by  $\uparrow_{i-1}^0(\nu)$ , where  $m$  is the greatest letter of  $\nu$ . The action of the symmetric group

$\mathfrak{S}_n$  satisfies  $E_\sigma \cdot v = E_{v^{-1} \circ \sigma}$  where  $\circ$  is the composition of permutations. The operads  $As^\rightarrow$  and  $As^\leftarrow$  are isomorphic through the linear map  $\phi : As^\rightarrow \rightarrow As^\leftarrow$  satisfying  $\phi(E_\sigma) = E_{\sigma^{-1}}$ .

- The *duplicial operad* [4] is the nonsymmetric operad  $\text{Dup}$  wherein for any  $n \geq 1$ ,  $\text{Dup}(n)$  is the linear span of the set  $\mathfrak{B}_n$ . The set  $\{E_t : t \in \mathfrak{B}_n, n \geq 1\}$  is a basis of  $\text{Dup}$ . The partial composition of  $\text{Dup}$  satisfies  $E_t \circ_i E_s = E_\tau$  where  $\tau$  is the binary tree obtained by replacing the  $i$ -th internal node  $u$  of  $t$  for the infix traversal by a copy of  $s$  and by grafting the left subtree of  $u$  on the leftmost leaf of the copy and by grafting the right subtree of  $u$  on the rightmost leaf of the copy. For instance, in  $\text{Dup}$ , we have

$$E \circ_6 E = E \quad (1.5)$$

- For any  $s \geq 1$ , the *s-interstice operad* [5, 6] is the nonsymmetric operad  $l_s$  wherein for any  $n \geq 1$ ,  $l_s(n)$  is the linear span of the set  $[s]^n$ . The set  $\{E_u : u \in [s]^n, n \geq 1\}$  is a basis of  $l_s$ . The partial composition of  $l_s$  satisfies

$$E_u \circ_i E_v := E_{u(1,i-1) \cdot v \cdot u(i,\ell(u))}. \quad (1.6)$$

For instance, in  $l_2$ , we have  $E_{121121} \circ_4 E_{21} = E_{121 \ 21 \ 121}$ .

As shown in [2],  $\text{Dup}$  and  $l_2$  are nonsymmetric operad quotients of  $PA_s^\rightarrow$ .

## 2 Operadic structures on packed words

We introduce two generalizations  $PA_s^\rightarrow$  and  $PA_s^\leftarrow$  respectively of  $As^\rightarrow$  and  $As^\leftarrow$  on the linear span of the set of packed words seen through two different graduations.

### 2.1 Right version

Let  $PA_s^\rightarrow$  be the graded space such that for any  $n \geq 1$ ,  $PA_s^\rightarrow(n)$  is the linear span of the set  $\mathfrak{P}_n$ . The set  $\{E_u : u \in \mathfrak{P}_n, n \geq 1\}$  is a basis of  $PA_s^\rightarrow$ . Let us endow  $PA_s^\rightarrow$  with the operations  $\circ_i$  defined, for any  $u \in \mathfrak{P}_n[r]$ ,  $i \in [n]$ , and  $v \in \mathfrak{P}_m[s]$ , by

$$E_u \circ_i E_v := E_{\uparrow_{s-1}^{u(i)}(u(1,i-1)) \cdot \uparrow_{u(i)-1}^0(v) \cdot \uparrow_{s-1}^{u(i)-1}(u(i+1,n))}. \quad (2.1)$$

For instance,  $E_{2311223} \circ_4 E_{221} = E_{2411 \ 332 \ 34}$ .

Intuitively, the partial composition  $E_u \circ_i E_v$  of  $\text{PAs}^{\rightarrow}$  is similar to the one  $As^{\rightarrow}$  but with the difference that the occurrences of  $u(i)$  in  $u$  having a position greater than  $i$  are incremented by  $s - 1$  where  $s$  is the maximal value of  $v$ . In terms of permutation matrices, we have

$$E \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \circ_4 E \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = E \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.2)$$

**Theorem 2.1.** *The graded space  $\text{PAs}^{\rightarrow}$ , endowed with the partial composition maps  $\circ_i$ , is a nonsymmetric unital operad.*

We call  $\text{PAs}^{\rightarrow}$  the *right associative operad of packed words*. Remark that in [10] and [18], other operads involving the same space of packed words are constructed. The operads  $\text{PAs}^{\rightarrow}$  and  $\text{PW}$ , which is defined in [10], are not isomorphic. Indeed, a simple inspection shows that any minimal generating set of  $\text{PW}$  contains two elements of arity 3 while any minimal generating set  $\text{PAs}^{\rightarrow}$  has no element of this arity.

For any  $u \in \mathfrak{P}_k$  and  $i \in [k]$ , the *record*  $\text{rec}_i(u)$  of  $u$  at  $i$  is  $\text{rec}_i(u) := \text{st}(u)(i)$ . For example, taking  $u = 2311223$  and  $i = 5$ , we have  $\text{st}(u) = 3612457$ , so  $\text{rec}_5(u) = \text{st}(u)(5) = 4$ . In what follows, we consider that the maps  $\text{rec}$  and  $\text{st}$  are extended linearly on the graded space  $\text{PAs}^{\rightarrow}$  through its E-basis. In the same way, we extend linearly the action  $\cdot$  of (1.2) on  $\text{PAs}^{\rightarrow}$  through its E-basis.

As shown by the next result, the operad  $\text{PAs}^{\rightarrow}$  relates well with  $As^{\rightarrow}$  and  $\mathfrak{l}_2$ .

**Theorem 2.2.** *Let  $n \in \mathbb{N}$ ,  $u \in \mathfrak{P}_n$ , and  $v \in \mathfrak{P}$ . For any  $i \in [n]$ ,*

$$E_u \circ_i E_v = \left( E_{\chi(u)} \circ_{\text{rec}_i(u)} E_{\chi(v)} \right) \cdot B_i(\text{st}(u), \text{st}(v)), \quad (2.3)$$

where the partial composition  $\circ_{\text{rec}_i(u)}$  is the one of  $\mathfrak{l}_2$ . In particular,  $\chi(E_u \circ_i E_v) = E_{\chi(u)} \circ_{\text{rec}_i(u)} E_{\chi(v)}$  and  $\text{st}(E_u \circ_i E_v) = E_{B_i(\text{st}(u), \text{st}(v))}$ .

Following with our example let  $u = 2311223$  and  $v = 221$ . Notice that  $\text{st}(u) = 3612457$ , so that  $\text{rec}_5(u) = 4$ . Also,  $\text{st}(v) = 231$ . Therefore,

$$\left( E_{\chi(u)} \circ_{\text{rec}_i(u)} E_{\chi(v)} \right) \cdot B_i(\text{st}(u), \text{st}(v)) = E_{112233344} \cdot 381256479 = E_{241133234}, \quad (2.4)$$

which agrees with our previous example at the beginning of this section.

## 2.2 Left version

Let  $\text{PAs}^{\leftarrow}$  be the graded space such that for any  $n \geq 1$ ,  $\text{PAs}^{\leftarrow}(n)$  is the linear span of the set  $\mathfrak{P}[n]$ . The set  $\{E_u : u \in \mathfrak{P}[n], n \geq 1\}$  is a basis of  $\text{PAs}^{\leftarrow}$ . Let us endow  $\text{PAs}^{\leftarrow}$  with the operations  $\circ_i$  defined, for any  $u \in \mathfrak{P}[n]$ ,  $i \in [n]$ , and  $v \in \mathfrak{P}[m]$ , by  $E_u \circ_i E_v = E_w$  where  $w$  is the word obtained by replacing in  $\uparrow_{m-1}^i(u)$  all occurrences of  $i$  by  $\uparrow_{i-1}^0(v)$ . For instance,

$$E_{231314} \circ_3 E_{122} = E_{2\ 344\ 1\ 344\ 15}. \quad (2.5)$$

In terms of permutation matrices, we have

$$E \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \circ_3 E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = E \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.6)$$

Let us also endow  $\text{PAs}^{\leftarrow}$  with the right action  $\cdot$  of the symmetric groups such that, for any  $u \in \mathfrak{P}[n]$  and  $\sigma \in \mathfrak{S}_n$ ,

$$E_u \cdot \sigma = E_{\sigma^{-1}(u(1)) \dots \sigma^{-1}(u(\ell(u)))}. \quad (2.7)$$

For instance,  $E_{1411232} \cdot 3142 = E_{2322414}$ .

**Theorem 2.3.** *The graded space  $\text{PAs}^{\leftarrow}$ , endowed with the partial composition maps  $\circ_i$  and the action  $\cdot$  of the symmetric groups, is a symmetric operad.*

We call  $\text{PAs}^{\leftarrow}$  the *left associative operad of packed words*. Observe that this operad is not combinatorial since for any  $n \geq 1$ , there are infinitely many packed words with  $n$  as maximal value. However, as we shall see in the next sections, this operad contains interesting quotient operads which are combinatorial.

## 3 Operadic quotients

In this section, we regard the operad  $\text{PAs}^{\leftarrow}$  as set-operads through their E-basis. This is possible since the composition of two elements of the E-basis produces an element of the E-basis. For this reason, we shall write here  $u$  instead of  $E_u$  for any  $u \in \mathfrak{P}$ .

### 3.1 Permutative congruences

Let  $P$  be a predicate on  $\mathbb{P}^* \times \mathbb{P} \times \mathbb{P} \times \mathbb{P}^*$ . From  $P$ , we define the binary relation  $\leftrightarrow$  on  $\mathbb{P}^*$  satisfying  $uabv \leftrightarrow ubav$  for any  $u, v \in \mathbb{P}^*$  and  $a, b \in \mathbb{P}$  such that  $P(u, a, b, v)$  holds. Let also  $\equiv_P$  be the reflexive, symmetric, and transitive closure of  $\leftrightarrow$ . The predicate  $P$  is

1. *compatible with relabeling* if for any  $u, v \in \mathbb{P}^*$  and  $a, b \in \mathbb{P}$ ,  $P(u, a, b, v)$  implies that  $P(f(u), f(a), f(b), f(v))$  where  $f : \mathbb{P} \rightarrow \mathbb{P}$  is any a strictly monotone map;
2. *compatible with subwords* if for any  $u, v \in \mathbb{P}^*$  and  $a, b \in \mathbb{P}$ ,  $P(u, a, b, v)$  implies that  $P(u', a, b, v')$  for any  $u', v' \in \mathbb{P}^*$  such that  $u$  is a subword of  $u'$  and  $v$  is a subword of  $v'$ .

Observe that when  $P$  is compatible with subwords,  $\equiv_P$  is a monoid congruence of the free monoid on  $\mathbb{P}$ . Here are some examples of predicates compatible with relabeling and subwords:

- Let  $P_C$  be the *commutative predicate*, satisfying  $P_C(u, a, b, v)$  for any  $u, v \in \mathbb{P}^*$  and  $a, b \in \mathbb{P}$ .
- Let  $P_S$  be the *sylvester predicate*, satisfying  $P_S(u, a, c, v)$  for any  $u, v \in \mathbb{P}^*$  and  $a, c \in \mathbb{P}$  such that there exists a letter  $b$  in  $v$  such that  $a \leq b < c$ . The equivalence relation  $\equiv_{P_S}$  is the *sylvester congruence* and has been introduced in [12] to provide an alternative construction of the Loday-Ronco Hopf algebra.
- Let  $P_H$  be the *hypoplactic predicate*, satisfying  $P_H(u, a, c, v)$  for any  $u, v \in \mathbb{P}^*$  and  $a, c \in \mathbb{P}$  such that there exists a letter  $b$  in  $u$  such that  $a < b \leq c$  or there exists a letter  $b$  in  $v$  such that  $a \leq b < c$ . The equivalence relation  $\equiv_{P_H}$  is the *hypoplactic congruence* and has been introduced in [14]. This equivalence relation can be used to provide a construction of the Hopf algebra **NCSF** of noncommutative symmetric functions.
- Let  $P_B$  be the *Baxter predicate*, satisfying  $P_B(u, a, c, v)$  for any  $u, v \in \mathbb{P}^*$  and  $a, c \in \mathbb{P}$  such that there exist a letter  $b$  in  $u$  and a letter  $b'$  in  $v$  such that  $a \leq b' < b \leq c$  or  $a < b \leq b' < c$ . The equivalence relation  $\equiv_{P_B}$  is the *Baxter congruence* and has been introduced in [9] to construct the Baxter Hopf algebra which contains the Loday-Ronco Hopf algebra as quotient.

In contrast, the *plactic predicate*  $P_P$ , satisfying  $P_P(u, a, c, v)$  for any  $u, v \in \mathbb{P}^*$  and  $a, c \in \mathbb{P}$  such that  $v$  is nonempty and its first letter  $b$  is such that  $a \leq b < c$  or  $u$  is nonempty and its last letter  $b$  is such that  $a < b \leq c$ , is not compatible with subwords. The equivalence relation  $\equiv_{P_P}$  enjoys a lot of properties (see for instance [7] for properties related to the construction of Hopf algebras) but does not play any role in this operadic context.

**Theorem 3.1.** *If  $P$  is a predicate compatible with relabeling and subwords, then  $\equiv_P$  is a non-symmetric operad congruence of  $\text{PAs}^{\leftarrow}$ .*

When  $P$  satisfies the prerequisites of Theorem 3.1,  $\equiv_P$  is a *permutative congruence*. Let us denote by  $\theta_P : \text{PAs}^{\leftarrow} \rightarrow \text{PAs}^{\leftarrow}$  the map sending any  $u \in \text{PAs}^{\leftarrow}$  to the minimal word w.r.t. the lexicographic order of the  $\equiv_P$ -equivalence class of  $u$ .



### 3.2 Quotients forming combinatorial operads

For any  $k \geq 1$ , let  $\text{first}_k : \text{PAs}^{\leftarrow} \rightarrow \text{PAs}^{\leftarrow}$  be the map sending any  $u \in \text{PAs}^{\leftarrow}$  to the packed word obtained by deleting any occurrence of a letter  $a$  provided that there are at least  $k$  occurrence of  $a$  on its left. For instance,  $\text{first}_2(421431412) = 4214312$ . Let us denote by  $\equiv_k$  the equivalence relation on  $\text{PAs}^{\leftarrow}$  satisfying, for any  $u, v \in \text{PAs}^{\leftarrow}$ ,  $u \equiv_k v$  if  $\text{first}_k(u) = \text{first}_k(v)$ .

**Proposition 3.2.** *For any  $k \geq 1$ , the equivalence relation  $\equiv_k$  is a nonsymmetric operad congruence of  $\text{PAs}^{\leftarrow}$ .*

Since for any  $k \geq 1$  and  $n \geq 1$ , there are finitely packed words having  $n$  as maximal value and where each letter appears at most  $k$  times, the operad  $\text{PAs}^{\leftarrow} / \equiv_k$  is combinatorial. The quotient  $\text{PAs}^{\leftarrow} / \equiv_1$  is the left associative operad of permutations. The sequence of dimensions of  $\text{PAs}^{\leftarrow} / \equiv_2$  begins with 2, 14, 222, 6384, 291720, 19445040 and forms Sequence [A105749](#) of [20].

A predicate  $P$  is *right-trivial* if for any  $u, v \in \mathbb{P}^*$  and  $a, b \in \mathbb{P}$ ,  $P(u, a, b, v)$  implies that  $P(u, a, b, \epsilon)$ .

**Theorem 3.3.** *If  $P$  is a right-trivial predicate compatible with relabeling and subwords, then the maps  $\text{first}_k$  and  $\theta_P$  commute for any  $k \geq 1$ .*

Let us denote by  $\equiv_{P,k}$  the equivalence relation on  $\text{PAs}^{\leftarrow}$  satisfying, for any  $u, v \in \text{PAs}^{\leftarrow}$ ,  $u \equiv_{P,k} v$  if  $\theta_P(\text{first}_k(u)) = \theta_P(\text{first}_k(v))$ . By Theorem 3.3,  $\equiv_{P,k}$  is a nonsymmetric operad congruence of  $\text{PAs}^{\leftarrow}$ . Let us review some quotients of  $\text{PAs}^{\leftarrow}$  obtained via such congruences:

- The commutative predicate  $P_C$  is right-trivial. The nonsymmetric operad  $\text{PAs}^{\leftarrow} / \equiv_{P_C,1}$  is the nonsymmetric associative operad, and for any  $k \geq 1$  and any  $n \geq 1$ , the dimension of  $\text{PAs}^{\leftarrow} / \equiv_{P_C,k}(n)$  is  $k^n$ . A set of representatives of this quotient is the set of weakly increasing packed words having  $n$  as maximal value and such that each maximal factor of equal letters is of length at most  $k$ .
- The sylvester predicate is not right-trivial, but by setting  $\bar{P}_S$  as the predicate satisfying  $\bar{P}_S(u, a, b, v)$  if and only if  $\bar{P}_S(v, a, b, u)$ ,  $\bar{P}_S$  is right-trivial. The sequence of the dimensions of  $\text{PAs}^{\leftarrow} / \equiv_{\bar{P}_S,1}$  begins with 1, 2, 5, 14, 132, 429 and forms Sequence [A000108](#) of [20] of Catalan numbers.

**Proposition 3.4.** *The nonsymmetric operad  $\text{PAs}^{\leftarrow} / \equiv_{\bar{P}_S,1}$  is isomorphic to the duplicial operad.*

The sequence of the dimensions of  $\text{PAs}^{\leftarrow} / \equiv_{P,2}$  begins with 2, 10, 66, 498, 4066, 34970 and forms Sequence [A027307](#) of [20]. The operads  $\text{PAs}^{\leftarrow} / \equiv_{\bar{P}_S,k}$  are generalizations of the duplicial operad.

## 4 Analogue of Dynkin idempotents to Zie

The weak order on permutations has several analogues for packed words. The *right weak order* for packed words is defined via the following covering relation: for  $u, v \in \mathfrak{P}_k$   $u \prec_r v$  if and only if  $v = u \cdot \tau$ , for some transposition  $\tau \in \mathfrak{S}_k$  and  $\text{inv}(v) = \text{inv}(u) + 1$ . Given  $u \in \mathfrak{P}$ , the *set of left-inversions*  $\text{Inv}_\ell(u)$  of  $u$  is the set of pairs  $(i, j)$  such that  $i < j$  and all appearances of  $i$  in  $u$  occur after all appearances of  $j$  in  $u$ . The *left weak order* of packed words is defined via the following covering relation: for  $u, v \in \mathfrak{P}[n]$ ,  $u \prec_\ell v$  if and only if  $v = \tau \cdot u$ , for some transposition  $\tau \in \mathfrak{S}_n$  and  $|\text{Inv}_\ell(v)| = |\text{Inv}_\ell(u)| + 1$ .

Consider the following graded bases for  $\text{PAs}^\rightarrow$ :

$$E_u = \sum_{v \prec_r u} F_v \quad \text{and} \quad F_u = \sum_{u \prec_\ell v} M_v. \quad (4.1)$$

The set  $\{M_u : u \in \mathfrak{P}_n, n \geq 0\}$  is a linear basis of the primitive space of the Hopf algebra  $\text{WQSym}$  of packed words (see, for example, [3]). Following the terminology of [1], we called this space the *space of Zie elements*. In the case of permutations, a *Lie element* is an element of the primitive space of  $\text{FQSym}$ , the Hopf algebra of permutations [17, 7].

It is straightforward to show that the  $F$ -basis satisfies the same internal operation in  $\text{PAs}^\rightarrow$  of the  $E$ -basis, as described in (2.1). Given  $u, v \in \mathfrak{P}$ , define a new operation on  $\text{PAs}^\rightarrow$  as

$$\{F_u, F_v\} := (M_{12} \circ_2 F_v) \circ_1 F_u. \quad (4.2)$$

For instance,  $\{F_{11}, F_{121}\} = ((F_{12} - F_{21}) \circ_2 F_{121}) \circ_1 F_{11} = F_{11232} - F_{33121}$ . Remark that this operation is not the induced Lie bracket of  $\text{WQSym}$ ; while  $\{F_1, F_1\} = F_{12} - F_{21}$ , we have  $[F_1, F_1] = 0$ .

Now, given  $k \geq 1$ , the *Dynkin idempotent*  $\frac{1}{k} \theta_k$  is defined as the unique element in  $\text{As}^\rightarrow(k)$  for which  $\theta_k \cdot a_1 a_2 a_3 \cdots a_k = [\cdots [[a_1, a_2], a_3], \cdots, a_k]$  is the  $k - 1$  left nested commutator bracket in the vector space  $\mathbb{K}\langle \mathbb{N} \rangle$ , for any word  $a_1 a_2 a_3 \cdots a_k \in \mathbb{K}\langle \mathbb{N} \rangle$ . Here, we consider the natural right action of permutations (as in (2.7)). It is shown in [2, Lem. 5.2] that we can express  $\theta_k$  as  $\theta_k = \{ \dots \{ \{ F_1, F_1 \}, F_1 \}, \dots, F_1 \}$  (there are  $k - 1$  left nested brackets). Given a composition  $c = c_1 c_2 \cdots c_n \in \mathbb{N}^*$  of sum  $k$ , we define the *c-Dynkin's idempotent*  $\frac{1}{k} \theta_k^c$ , where

$$\theta_k^c := \left\{ \dots \left\{ \left\{ F_{\text{id}_{c_1}}, F_{\text{id}_{c_2}} \right\}, F_{\text{id}_{c_3}} \right\}, \dots, F_{\text{id}_{c_n}} \right\} \quad (4.3)$$

and  $\text{id}_r$  denotes the packed word  $11 \cdots 1$  of length  $r$ . More generally, let  $\text{id}(c) \in \mathfrak{P}_k[n]$  be the non-decreasing packed word with  $c_r$  copies of  $r$ , for every  $r \in [n]$ .

**Theorem 4.1.** *For any  $k \geq 1$ ,*

$$\theta_k^c = \sum_{\substack{\text{id}(c) \leq_\ell u \\ u = \text{id}(c_1) 2 \circ_2 u'}} M_u. \quad (4.4)$$

*In particular,  $\theta_k^c$  is a Zie element in  $\text{WQSym}$ .*

## 5 Open questions and future work

Here are some questions and possible ways to continue this work.

- (1) **Bases of  $\text{PAs}^{\rightarrow}$  and partial orders on packed words** — One of the initial motivations of this work was to mimic the constructions relating the associative operad and the weak order developed by Aguiar and Livernet in [2]. They showed that the partial composition on the M-basis of  $\text{As}$  is encoded by intervals, which is not the case for  $\text{PAs}^{\rightarrow}$ . This leads to the exploration of other posets structures on packed words.
- (2) **Quotients of  $\text{PAs}^{\rightarrow}$**  — A second motivation of this work was to construct at the level of our operads of packed words similar structures as the duplicital and interstice operads, which are quotients of  $\text{As}$  [2]. This has been done for  $\text{PAs}^{\leftarrow}$ . The question remains open for  $\text{PAs}^{\rightarrow}$ . We conjecture that the analog of the duplicital operad, as quotient of  $\text{PAs}^{\rightarrow}$ , involves Schröder trees. There are other operads on Schröder trees defined in [5] and [10]. A question is to investigate whether these operads are isomorphic.
- (3) **Generalizations of  $\text{PAs}^{\rightarrow}$**  — The partial compositions of the  $\text{PAs}^{\rightarrow}$  can be extended on generalizations of packed words, while preserving the structure of a nonsymmetric operad. Such extensions work both for the sets of parking functions and of endofunctions. The study of these two new structures is planned. In contrast, the similar extensions for  $\text{PAs}^{\leftarrow}$  do not form operads.
- (4) **Quotients of  $\text{PAs}^{\leftarrow}$**  — The study of the quotients  $\text{PAs}^{\leftarrow} / \equiv_{P_C, k}$  and  $\text{PAs}^{\leftarrow} / \equiv_{\bar{P}_S, k}$  of  $\text{PAs}^{\leftarrow}$  is planned. This last quotient involves Schröder trees subjected to some conditions and with a unusual notion of arity. This further study of operads encompasses the description of presentations by generators and relations and combinatorial descriptions of their partial compositions on right combinatorial objects.
- (5) **Diagonal of the permutahedron** — Given a polytope  $P$ , it is a general problem to find a cellular approximation which is homotopic to the diagonal map  $\Delta_P : P \rightarrow P \times P$ , and which agrees with  $\Delta_P$  on the vertices of  $P$ . For instance, the cellular chains on the associahedra are endowed with an operad structure which encodes associative algebras up to homotopy. A natural question is how to relate our two operad structures  $\text{PAs}^{\leftarrow}$  and  $\text{PAs}^{\rightarrow}$  to coherent cellular approximations of the diagonal of the permutahedra.

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